XVIII. *Researches on the Geometrical Properties of Elliptic Integrals.*

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SECTION I.

I. IN placing before the Royal Society the following researches on the geometrical types of elliptic integrals, which nearly complete my investigations on this interesting subject, I may be permitted briefly to advert to what had already been effected in this department of geometrical research. LEGENDRE, to whom this important branch of mathematical science owes so much, devised a plane curve, whose rectification might be effected by an elliptic integral of the first order. Since that time many other geometers have followed his example, in contriving similar curves, to represent, either by their quadrature or rectification, elliptic functions. Of those who have been most successful in devising curves which should possess the required properties, may be mentioned M. GUDERMANN, M. VERHULST of Brussels, and M. SERRET of Paris. These geometers however have succeeded in deriving from those curves scarcely any of the properties of elliptic integrals, even the most elementary. This barrenness in results was doubtless owing to the very artificial character of the genesis of those curves, devised, as they were, solely to satisfy one condition only of the general problem*.

In 1841 a step was taken in the right direction. MM. CATALAN and GUDERMANN, in the journals of Liouville and Crelle, showed how the arcs of spherical conic sections might be represented by elliptic integrals of the third order and *circular* form. They did not, however, extend their investigations to the case of elliptic integrals of the third order and logarithmic form; nor even to that of the first order. These cases still remained, without any analogous geometrical representative, a blemish to the theory.

Some years ago, when engaged in the discussion of the problem of the rotation of a rigid body round a fixed point, by the help of an auxiliary ellipsoid, I had continually brought under my notice, in the course of my investigations, the sections of a sphere by a concentric cone, or as they now are generally named, spherical conic

* LEGENDRE a cherché à représenter en général, la fonction dig. (c, ϕ) par un arc de courbe; mais ses tentatives ne nous ont pas semblé heureuses, car il n'est parvenu à resoudre completément le problème, qu'en employant une courbe transcendante, dans laquelle l'amplitude ϕ et l'arcs ont entre eux une relation géométrique encore plus difficile à saisir que dans la lemniscate.—VERHULST, *Traité des Fonctions Elliptiques*, p. 295.

sections. It accordingly became necessary that I should give especial attention to the nature of those curves. I succeeded in showing that the elliptic integral of the first order, which is merely a particular case of the circular form of elliptic integrals of the third order, represents a spherical conic section whose principal arcs have a certain relation to each other. Besides, I was so fortunate as to hit upon the true geometrical representative of an elliptic integral of the third order and logarithmic form. I discovered it to be the curve of intersection of a right elliptic cylinder by a paraboloid of revolution having its axis coincident with that of the cylinder. These researches were published in the early part of the present year*. There still remained, without investigation, the case when the parameter is negative and greater than 1. The geometrical representative of this peculiar form, I announced to be a curve, which I called the *Logarithmic hyperbola.* In the Theory of Elliptic Integrals, p. 159, I have said, "If a right cylinder standing on a plane hyperbola as a base, be substituted for the elliptic cylinder, the curve of intersection may be named the *logarithmic hyperbola.* It will have four infinite branches, whose asymptots will be the infinite arcs of two equal plane parabolas. This curve, and not the spherical ellipse, is the true analogue of the common hyperbola." No demonstration, however, of these properties was given in that treatise.

The main object of the following paper is to prove, that *Elliptic Integrals of every order, the parameter taking any value whatever between positive and negative infinity, represent the intersections of surfaces of the second order.*

To these curves may be given the appropriate name of *Hyperconic sections.*

These surfaces divide themselves into two classes, of which the sphere and the paraboloid of revolution are the respective types; from the one arise the circular functions, from the other the logarithmic and exponential. The circular integral of the third order is derived from the sphere, while the logarithmic function of the same order is founded on the paraboloid of revolution.

Although in the following pages I have, for the sake of simplicity, derived the properties of those curves, or of the integrals which represent them, from the intersections of these normal surfaces,—the sphere and the paraboloid,—with certain cylindrical surfaces; yet the intersections so produced may be considered as the intersections of these normal surfaces with various other surfaces of the second order. Let $U=0$ be the equation of the sphere or paraboloid, and $V=0$ the equation of the cylinder. The simultaneous equations $U=0$, $V=0$ give the equations of the curve of intersection. Let f be any abstract number whatever; then $U+fV=0$ is the equation of another surface of the second order passing through the curve of intersection. Let $U=0$ be the equation of a sphere, for example. Accordingly as we assign suitable values to the number f, we may make the equation $U+fV=0$ represent any central surface of the second order. But we cannot, by any substitution or

* The Theory of Elliptic Integrals, and the Properties of Surfaces of the Second Order, applied to the investigation of the motion of a body round a fixed point. London: G. BELL, 1851.

rational transformation, make the equation $U+fV=0$ represent a non-central surface instead of a central one, or *vice versâ*.

Although a remarkable relation exists between the areas and lengths of some of these hyperconics, such as the circle and the spherical ellipse, yet more distinctly to show the analogy which pervades all those curves, I have not had recourse in any case to the method of "elliptic quadratures," as it is termed*. We cannot admit such a violation of the law of geometrical continuity as to suppose, that while a function in one state represents a curve line, in another, immediately succeeding, it must express an area. Such can only be taken as a conventional explanation, until the real one, characterized by the simplicity of truth, shall present itself.

In the course of these investigations, it will be shown that the formulæ for the comparison of elliptic integrals, which are given by Legendre and other writers on this subject, follow simply as geometrical inferences from the fundamental properties of those curves; and that the ordinary conic sections are merely particular cases of those more general curves above referred to, under the name of hyperconic sections.

It will doubtless appear not a little singular, that the principal properties of those functions, their classification, their transformations, the comparison of integrals of the third order, with conjugate or reciprocal parameters, were all investigated and developed before geometers had any idea of the true geometrical origin of those functions. It is as if the formulæ of trigonometry had been derived from an algebraical definition, before the geometrical conception of the circle had been admitted. As trigonometry may be defined, the development of the properties of circular arcs, whether described on a plane or on the surface of a sphere; so this higher trigonometry, or the theory of elliptic integrals, may best be interpreted as the development of the relations which exist between the arcs of hyperconic sections.

Indeed it may with truth be asserted, that nearly all the principal functions, on which the resources of analysis have chiefly been exhausted, whether they be circular, logarithmic, exponential or elliptic, arise out of the solution of this one general problem, to determine the length of an arc of a hyperconic section.

It may be said, we cannot by this method derive any properties of elliptic integrals which may not algebraically be deduced from the fundamental expressions appropriately assumed. But surely no one will assert that the properties of curve lines should be algebraically developed, without any reference to their geometrical types.

We might from algebraical expressions suitably chosen, derive every known property of curve lines, without having in any instance a conception of the geometrical types

* En considérant les fonctions elliptiques comme des secteurs, dont l'angle est précisément égal à l'amplitude ϕ, nous avons en l'avantage de justifier la dénomination d'amplitude appliquée à l'angle ϕ; et même celle de *fonctions elliptiques*, en général, puisque les courbes algébraiques par lesquelles nous avons représentés ces transcendantes, se construisent avec facilité au moyen des rayons vecteurs d'une ou de deux ellipses données. —Verhulst, *Traité des Fonctions Elliptiques*, p. 295.

which they represent. The theory of elliptic integrals was developed by a method the inverse of that pursued in establishing the formulæ of common trigonometry. In the latter case, the geometrical type was given—the circle—to determine the algebraical relations of its arcs. In the theory of elliptic integrals, the relations of the arcs of unknown curves are given, to determine the curves themselves. This is briefly the object of the present paper.

The true geometrical basis of this theory would doubtless long since have been developed, had not geometers sought to discover the types of those functions among plane curves. They were beguiled into this course by observing, that in one case—that of the second order—the representative curve is obviously a plane ellipse. Hence they were led by a seeming analogy to search for the types of the other integrals among plane curves also.

The author hopes in a future communication to the Royal Society, the present having grown under his hands beyond the limits he anticipated, to extend his researches to elliptic integrals with imaginary parameters, and to show the true geometrical meaning of such expressions. It has long been known, that, by the aid of the imaginary transformation $\sin\phi=\sqrt{-1}\tan\psi$, we may pass from the logarithmic to the circular type, and conversely; but it has not, however, been observed that this transformation enables us to effect this transition, because it changes the algebraic expression for the arc of a parabola into that for a circular arc or area, and conversely. The striking analogies developed between the formulæ of the trigonometry of the circle and that of the parabola will be found very curious and instructive.

I have attempted thus to place on its true geometrical basis, a somewhat abstruse department of analysis, and to clear up the elementary notions from which it may, with the utmost simplicity, be developed. It is only in the maturity of a science, that the relations which bind together its cardinal ideas become simplified. An author, who has himself contributed much to the progress of mathematical science, well observes,—" qui il est bien rare qu'une théorie sorte sous sa forme la plus simple des mains de son premier auteur. Nous pensons qu'on sert peut-être plus encore la science en simplifiant, de la sorte, des théories déjà connues, qu'en l'enrichissant de théories nouvelles, et c'est là un sujet auquel on ne saurait s'appliquer avec trop de soin."—GERGONNE, *Annales des Mathématiques*, tom. xix. p. 338.

II. I have ventured to make some alterations in the established notation of elliptic integrals. I have written i for the modulus, instead of c; and j for its complement instead of b; so that $i^2+j^2=1$.

The symbol c, used by writers on this subject to designate the modulus, was adopted by analogy from the formula for the rectification of a plane elliptic arc by an integral of the second order. Although in the circular forms of the third order it still signifies a certain ellipticity, yet it has no longer the same signification in the usual form of the first order, or in the logarithmic form of the third.

Instead of the usual symbol, $\Delta=\sqrt{1-c^2\sin^2\varphi}=\sqrt{1-i^2\sin^2\varphi}$, $\sqrt{\mathrm{I}}$ has been substituted, when i is the modulus. Should it become necessary to designate the amplitude, the expression may be written $\sqrt{\mathrm{I}_\varphi}$, or $\sqrt{{}_\varphi\mathrm{I}}$.

For the elliptic integrals of the first and second orders, which are usually written $F_c(\varphi)$ and $E_c(\varphi)$, I have substituted $\int\frac{d\varphi}{\sqrt{\mathrm{I}}}$ and $\int d\varphi\sqrt{\mathrm{I}}$. The surface of revolution may be named the *generating surface*, while the intersecting surface is always a cylindrical surface. The parameter, of which p is the general symbol, we shall suppose to vary from positive to negative infinity, and to pass through all intermediate states of magnitude.

The nature of the representative curve will depend on the value assigned to the parameter p in the expression $\mathrm{K}\int\frac{d\varphi}{[\pm p\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}$. The modulus we shall assume to be invariable and less than 1. In this progress from $+\infty$ to $-\infty$, the parameter passes through thirteen distinct values, each of which will cause a variation in the species or properties of the hyperconic section, the representative curve of the given elliptic integral.

In the following Table we may observe that the generating surface in passing from a sphere to a paraboloid, in its state of transition, becomes a plane.

It is somewhat remarkable, that the common form of the elliptic integral of the first order does not appear in the Table, although it is implicitly contained in cases II. and VIII.; for in the circular form of the third order, when the parameter is equal to the modulus i, we can reduce the third order to the first. The reason why the first form of elliptic integral does not appear in the Table is this; in the thirteen cases given, the origin is placed at the centre, or symmetrically with respect to the represented curve. When the elliptic integral of the first order is given in the usual form, without a parameter, it represents a spherical parabola, but the origin is non-symmetrical, that is, the origin is placed at a focus. See Theory of Elliptical Integrals, p. 33.

Instead of p, the general symbol for the parameter, we may substitute for it particular values, such as l, m, or n, as the case may require. The quantities l, m, n, i and j, are connected by the following equations:—

$$\left.\begin{array}{l} i^2+j^2=1,\quad lm=i^2,\quad \text{and } m-n+mn=i^2, \text{ in the circular form,} \\ i^2+j^2=1,\quad ln=i^2,\quad \text{and } m+n-mn=i^2, \text{ in the logarithmic form,} \end{array}\right\} \quad \ldots \quad (1.)$$

m and n may be called *conjugate parameters*; while l and m, or l and n may be termed *reciprocal parameters*.

These thirteen cases are exhibited in the following Table.

TABLE.

Case.	Sign.	Parameter.	Generating surface.	Cylindrical surface.	Hyperconic section.
I.	$+$	$p=n=\infty$	Sphere	Elliptic cylinder	Circular sections of elliptic cylinder.
II.	$+$	$p=n=i$, or $m=n$.	Sphere	Elliptic cylinder	Spherical parabola.
III.	$+$	$p=n>0$......	Sphere	Elliptic cylinder	Spherical ellipse.
IV.	$\pm$	$p=n=0$......	Plane	Elliptic cylinder	Plane ellipse.
V.	$-$	$p=m=1-\sqrt{1-i^2}$, or $m=n$.	Paraboloid indefinitely attenuated.	Circular cylinder.........	Circular logarithmic ellipse.
VI.	$-$	$p=m$, or $p=n<i^2$ ...	Paraboloid	Elliptic cylinder	Logarithmic ellipse.
VII.	$-$	$p=m=i^2$......	Plane	Elliptic cylinder	Plane ellipse.
VIII.	$-$	$p=m=i$......	Sphere	Elliptic cylinder	Spherical parabola.
IX.	$-$	$p=m>i^2$ $p=m<1$.	Sphere	Elliptic cylinder	Spherical ellipse.
X.	$-$	$p=l=1$......	Plane	Hyperbolic cylinder ...	Plane hyperbola.
XI.	$-$	$p=l>1$......	Paraboloid	Hyperbolic cylinder ...	Logarithmic hyperbola.
XII.	$-$	$p=l=1+\sqrt{1-i^2}$, or $m=n$.	Paraboloid	Hyperbolic cylinder.	Equiparametral logarithmic hyperbola.
XIII.	$-$	$p=l=\infty$	Paraboloid	Vertical plane	Parabola.

Cases I., IV., VII., X., XIII. give the formulæ for the rectification of the ordinary conic sections; the generating surface in these cases being a plane. When the generating surface is a sphere, we get the spherical hyperconic sections; when a paraboloid, the logarithmic hyperconic sections result.

SECTION II.—*On the Spherical Ellipse.*

III. A spherical ellipse may be defined as the curve of intersection of a cone of the second degree with a concentric sphere.

In the spherical ellipse there are two points analogous to the foci of the plane ellipse, such *that the sum of the arcs of the great circles drawn from those points to any point on the curve is constant.* Let α and β be the principal semiangles of the cone; 2α and 2β are therefore the *principal arcs* of the spherical ellipse. Let two right lines be drawn from the vertex of the cone in the plane of the angle 2α, making with the internal axe of the cone equal angles ε, such that

$$\cos\varepsilon=\frac{\cos\alpha}{\cos\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

These lines are usually called *focals*, or the *focal lines* of the cone. The points in

which they meet the surface of the sphere are termed the *foci* of the spherical ellipse.

IV. *Every umbilical surface of the second order has two concentric circular sections, whose planes, in the case of cones, pass through the greater of the external axes. Perpendiculars drawn to the planes of those sections, passing through the vertex,—they may be called the* CYCLIC AXES *of the cone—make with the internal axis of the cone in the plane of* 2β*—the plane passing through the internal and the lesser external axis*—equal angles η, such that

$$\cos\eta=\frac{\sin\beta}{\sin\alpha}. \quad \text{(3.)}$$

Let a series of planes be drawn through the vertex, and perpendicular to the successive sides of the cone. This series of planes will envelope a second cone, which usually is called the *supplemental cone* to the former. The cones are so related, that the planes of the circular sections of the one are perpendicular to the focals of the other, and conversely.

V. The equation of the spherical ellipse may be found as follows, from simple geometrical considerations.

Let 2α and 2β be the greatest and least vertical angles of the cone; the origin of coordinates being placed at the common centre of the sphere and cone. Let the internal axis of the cone meet the surface of the sphere in the point Z, which may be taken as the pole. Let ρ be an arc of a great circle drawn from the point Z to any point Q on the curve. ψ being the angle which the plane of this circle makes with the plane of 2α, we shall have for the polar equation of the spherical ellipse,

$$\frac{1}{\tan^2\rho}=\frac{\cos^2\psi}{\tan^2\alpha}+\frac{\sin^2\psi}{\tan^2\beta}.$$

To show this, through the point Z let a tangent plane be drawn to the sphere. This plane will intersect the cone in an ellipse. This ellipse may be called the *plane base* of the cone, while the portion of the surface of the sphere within the cone may be termed the *spherical base* of the cone. The plane of the great circle passing through Z and Q will cut the plane base of the cone in the radius vector R; and if we write A and B for the semiaxes of this ellipse, whose plane touches the sphere, we shall have, for the common polar equation of this ellipse, the centre being the pole,

$$\frac{1}{R^2}=\frac{\cos^2\psi}{A^2}+\frac{\sin^2\psi}{B^2}.$$

Now the radius of the sphere being k, and ρ, α, β, the angles subtended at the centre by R, A, B, we shall clearly have

$$R=k\tan\rho, \qquad A=k\tan\alpha, \qquad B=k\tan\beta; \quad \text{(4.)}$$

whence

$$\frac{1}{\tan^2\rho}=\frac{\cos^2\psi}{\tan^2\alpha}+\frac{\sin^2\psi}{\tan^2\beta}. \quad \text{(5.)}$$

We may write this equation in the form

$$\frac{1-\sin^2\rho}{\sin^2\rho}=\frac{\cos^2\psi}{\sin^2\alpha}(1-\sin^2\alpha)+\frac{\sin^2\psi}{\sin^2\beta}(1-\sin^2\beta);$$

or reducing,

$$\frac{1}{\sin^2\rho}=\frac{\cos^2\psi}{\sin^2\alpha}+\frac{\sin^2\psi}{\sin^2\beta}. \quad \text{(6.)}$$

This is the equation of the spherical ellipse under another form, which may be obtained independently, by orthogonally projecting the spherical ellipse on the plane of the external axes; or by taking the spherical ellipse as the symmetrical intersection of a right elliptic cylinder with the sphere.

VI. *If in the major principal arc 2α of the spherical ellipse, we assume two points equidistant from the centre, the distance ε being determined by the condition* $\cos\varepsilon=\frac{\cos\alpha}{\cos\beta}$, *as in* (2.), *the sum of the arcs of the great circles drawn from these points —the foci—to any point on the spherical ellipse is constant, and equal to the principal arc 2α.* For a proof of this well-known property, the reader is referred to the Theory of Elliptical Integrals, p. 12.

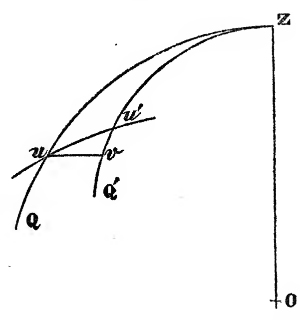
VII. *The product of the sines of the perpendicular arcs let fall from the foci of a spherical ellipse on the arc of a great circle touching it, is constant.*

Let ϖ and ϖ' be the perpendicular arcs let fall from the foci on the tangent arc of a great circle; we shall have

$$\sin\varpi \sin\varpi'=\sin(\alpha+\varepsilon)\sin(\alpha-\varepsilon). \quad \text{(7.)*}$$

VIII. To find an expression for the length of a curve described on the surface of a sphere, whose radius is 1.

Fig. 1.



Let u and u' be two consecutive points on the curve, ZQ, ZQ′ the arcs of two great circles passing through them inclined to each other at the indefinitely small angle $d\psi$. Through u let a plane be drawn perpendicular to OZ, and meeting the great circle ZQ′ in v.

Then ultimately uvu' may be taken as a right-angled triangle, whence $\overline{uu'}^2=\overline{uv}^2+\overline{u'v}^2$.

Now $uu'=d\sigma$, $uv=\sin\rho\, d\psi$, $u'v=d\rho$, whence

$$d\sigma=[d\rho^2+\sin^2\rho\, d\psi^2]^{\frac{1}{2}}. \quad \text{(8.)}$$

Integrating this expression between the limits $\rho_{\prime}$ and $\rho_{\prime\prime}$, or ψ and 0, accordingly as we take ρ or ψ for the independent variable, we get

$$\sigma=\int_{\rho''}^{\rho_{\prime}} d\rho\left[1+\sin^2\rho\left(\frac{d\psi}{d\rho}\right)^2\right]^{\frac{1}{2}}; \quad \text{or } \sigma=\int_0^{\psi} d\psi\left[\left(\frac{d\rho}{d\psi}\right)^2+\sin^2\rho\right]^{\frac{1}{2}}. \quad \text{(9.)}$$

IX. To apply this expression to find the length of an arc of a spherical ellipse.

In this case it will be found simpler to integrate the differential expression for an

* Theory of Elliptical Integrals, &c., p. 13.

arc of a curve, taking ρ instead of ψ as the independent variable. We may derive from (6.) the following expressions,

$$\sin^2\psi=\frac{\sin^2\beta}{\sin^2\rho}\left\{\frac{\sin^2\alpha-\sin^2\rho}{\sin^2\alpha-\sin^2\beta}\right\},\ \cos^2\psi=\frac{\sin^2\alpha}{\sin^2\rho}\left\{\frac{\sin^2\rho-\sin^2\beta}{\sin^2\alpha-\sin^2\beta}\right\}\ .\ .\ .\ .\ .\ . \quad (10.)$$

Differentiating the former with respect to ψ and ρ, and eliminating $\sin\psi$, $\cos\psi$; using for this purpose the relations established in (10.), we find

$$\frac{d\psi}{d\rho}=\frac{-\sin\alpha\sin\beta\cos\rho}{\sin\rho\sqrt{\sin^2\alpha-\sin^2\rho}\sqrt{\sin^2\rho-\sin^2\beta}}\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . \quad (11.)$$

Substituting this value of $\frac{d\psi}{d\rho}$ in the general expression for the arc; the resulting equation will become

$$\sigma=\int d\rho\left[\frac{\sin\rho\sqrt{\cos^2\rho-\cos^2\alpha\cos^2\beta}}{\sqrt{(\sin^2\alpha-\sin^2\rho)(\sin^2\rho-\sin^2\beta)}}\right],\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . \quad (12.)$$

an elliptic integral which may be reduced to the usual form by the following transformation: assume—

$$\cos^2\rho=\frac{\sin^2\alpha\cos^2\phi+\sin^2\beta\sin^2\phi}{\tan^2\alpha\cos^2\phi+\tan^2\beta\sin^2\phi}.\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . \quad (13.)$$

The limits of integration are 0 and $\frac{\pi}{2}$. Differentiating this expression, and introducing into (12.) the relations assumed in (13.), we obtain for the arc the following expression:—

$$\sigma=\frac{\tan\beta}{\tan\alpha}\sin\beta\int\left[\frac{d\phi}{\left[1-\left(\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}\right)\sin^2\phi\right]\sqrt{1-\left(\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}\right)\sin^2\phi}}\right]\ .\ .\ .\ . \quad (14.)$$

Let e be the eccentricity of the plane base of the cone, whose semiaxes are A and B, as in (V.),

$$e^2=\frac{A^2-B^2}{A^2}=\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha},\quad \text{as in (4.)},$$

(3.) gives $$\sin^2\eta=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha},$$

and we derive from (2.) $$\sin^2\varepsilon=\frac{\sin^2\alpha-\sin^2\beta}{\cos^2\beta};$$

or grouping these results together,

$$e^2=\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}=m$$

$$\sin^2\eta=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}=i^2\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . \quad (15.)$$

$$\sin^2\varepsilon=\frac{\sin^2\alpha-\sin^2\beta}{\cos^2\beta}=n,$$

If we introduce these values into (14.), the transformed equation will become

$$\sigma=\frac{\tan\beta}{\tan\alpha}\sin\beta\int\left[\frac{d\phi}{[1-e^2\sin^2\phi]\sqrt{1-\sin^2\eta\sin^2\phi}}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.)$$

an elliptic integral of the third order and circular form, since e^2 is greater than $\sin^2\eta$, and less than 1.

This is case IX. in the Table, page 6.

This is one of the simplest forms to which the rectification of an arc of a spherical ellipse can be reduced. The parameter of the elliptic integral is the square of the eccentricity of the plane elliptic base, and the modulus is the sine of half the angle between the planes of the circular sections of the cone.

If we write m for e^2, i for $\sin\eta$, and express the coefficient $\frac{\tan\beta}{\tan\alpha}\sin\beta$ in terms of m and i, the expression (16.) may be transformed into

$$\sigma=\left(\frac{1-m}{m}\right)\sqrt{mn}\int\left[\frac{d\phi}{[1-m\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.)$$

It is easily shown that the coefficient $\frac{\tan\beta}{\tan\alpha}\sin\beta$ of the elliptic integral in (16.) or its equal $\left(\frac{1-m}{m}\right)\sqrt{mn}$ is the square root of the *criterion of sphericity*,

$$\varkappa=(1-m)\left(1-\frac{i^2}{m}\right).$$

For if we substitute in this expression for i, its value given in (1.) $m-n+mn=i^2$, we shall find

$$\sqrt{\varkappa}=\frac{\tan\beta}{\tan\alpha}\sin\beta=\left(\frac{1-m}{m}\right)\sqrt{mn}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.)$$

As $\sqrt{\varkappa}$ is manifestly real, the elliptic integral is of the circular form.

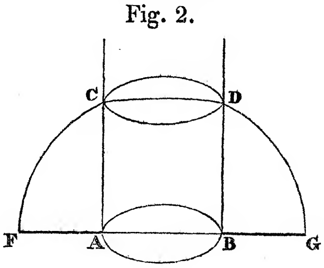
X. We may, by the method of rectangular coordinates, derive an expression for the arc of a spherical ellipse.

In this case we shall consider the spherical ellipse as the curve of intersection of a right elliptic cylinder by a sphere having its centre on the axis of the cylinder.

Let $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \text{and } x^2+y^2+z^2=k^2 \quad . \quad . \quad . \quad (19.)$$

be the equations of the cylinder and sphere, ABCD and FGCD, then $d\sigma$ being the element of an arc on the surface of a sphere whose radius is 1, $kd\sigma$ will be the element of the corresponding arc on the surface of the sphere whose radius is k.

Fig. 2.



Hence $$k\frac{d\sigma}{d\lambda}=\sqrt{\left(\frac{dx}{d\lambda}\right)^2+\left(\frac{dy}{d\lambda}\right)^2+\left(\frac{dz}{d\lambda}\right)^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20.)$$

x, y and z being functions of the independent variable λ.

Assume
$$\left.\begin{aligned} x^2&=\frac{a^4\cos^2\lambda}{a^2\cos^2\lambda+b^2\sin^2\lambda}, \quad y^2=\frac{b^4\sin^2\lambda}{a^2\cos^2\lambda+b^2\sin^2\lambda} \\ z^2&=\frac{a^2(k^2-a^2)\cos^2\lambda+b^2(k^2-b^2)\sin^2\lambda}{a^2\cos^2\lambda+b^2\sin^2\lambda}. \end{aligned}\right\} \quad \ldots \quad (21.)$$

Differentiating these expressions,

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)^2=\frac{a^4b^4\sin^2\lambda}{[a^2\cos^2\lambda+b^2\sin^2\lambda]^3}, \quad \left(\frac{\mathrm{d}y}{\mathrm{d}\lambda}\right)^2=\frac{a^4b^4\cos^2\lambda}{[a^2\cos^2\lambda+b^2\sin^2\lambda]^3};$$

and as $x\mathrm{d}x+y\mathrm{d}y+z\mathrm{d}z=0$,

$$\left(\frac{\mathrm{d}z}{\mathrm{d}\lambda}\right)^2=\frac{a^4b^4(a^2-b^2)^2\sin^2\lambda\cos^2\lambda}{[a^2\cos^2\lambda+b^2\sin^2\lambda]^3[a^2(k^2-a^2)\cos^2\lambda+b^2(k^2-b^2)\sin^2\lambda]}. \quad \ldots \quad (22.)$$

Substituting these expressions in (20.), we find

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}\right)^2=\frac{a^4b^4[a^2(k^2-a^2)\cos^2\lambda+b^2(k^2-b^2)\sin^2\lambda+(a^2-b^2)^2\sin^2\lambda\cos^2\lambda]}{k^2[a^2\cos^2\lambda+b^2\sin^2\lambda]^3[a^2(k^2-a^2)\cos^2\lambda+b^2(k^2-b^2)\sin^2\lambda]}. \quad \ldots \quad (23.)$$

The numerator of this expression may be resolved into the factors

$$[a^2\cos^2\lambda+b^2\sin^2\lambda][(k^2-a^2)\cos^2\lambda+(k^2-b^2)\sin^2\lambda],$$

and the equation may now be written

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}=\frac{a^2b^2\sqrt{(k^2-a^2)\cos^2\lambda+(k^2-b^2)\sin^2\lambda}}{k[a^2\cos^2\lambda+b^2\sin^2\lambda]\sqrt{a^2(k^2-a^2)\cos^2\lambda+b^2(k^2-b^2)\sin^2\lambda}}. \quad \ldots \quad (24.)$$

Assume
$$\tan^2\psi=\left(\frac{k^2-b^2}{k^2-a^2}\right)\tan^2\lambda. \quad \ldots \quad (25.)$$

Hence
$$\frac{\mathrm{d}\lambda}{\mathrm{d}\psi}=\frac{\sqrt{(k^2-a^2)(k^2-b^2)}}{(k^2-a^2)\sin^2\psi+(k^2-b^2)\cos^2\psi}.$$

(24.) may now be transformed into

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\psi}=\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}\psi}=\frac{a^2b^2\sqrt{(k^2-a^2)(k^2-b^2)}}{k[a^2(k^2-b^2)\cos^2\psi+b^2(k^2-a^2)\sin^2\psi]\sqrt{a^2\cos^2\psi+b^2\sin^2\psi}}. \quad \ldots \quad (26.)$$

If we imagine a concentric cone to pass through the mutual intersection of the cylinder and the sphere, we shall have

$$\left.\begin{aligned} a=k\sin\alpha, \quad & b=k\sin\beta, \\ \sin^2\eta=\frac{a^2-b^2}{a^2}, \quad & e^2=\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}=\frac{k^2(a^2-b^2)}{a^2(k^2-b^2)} \end{aligned}\right\} \quad \ldots \quad (27.)$$

Whence (26.) may be transformed into

$$\sigma=\frac{\tan\beta}{\tan\alpha}\sin\beta\int\left[\frac{\mathrm{d}\psi}{[1-e^2\sin^2\psi]\sqrt{1-\sin^2\eta\sin^2\psi}}\right], \quad \ldots \quad (28.)$$

an expression identically the same with (16.).

The angle ψ in this expression is identical with ϕ in (16.).

For
$$x^2+y^2=\frac{a^4\cos^2\lambda+b^4\sin^2\lambda}{a^2\cos^2\lambda+b^2\sin^2\lambda}=\frac{a^4+b^4\tan^2\lambda}{a^2+b^2\tan^2\lambda};$$

eliminating $\tan\lambda$ by (25.),

$$x^2+y^2=\frac{a^4(k^2-b^2)\cos^2\psi+b^4(k^2-a^2)\sin^2\psi}{a^2(k^2-b^2)\cos^2\psi+b^2(k^2-a^2)\sin^2\psi}.$$

Now $a^2=k^2\sin\alpha$, $b^2=k^2\sin^2\beta$, $k^2-a^2=k^2\cos^2\alpha$, $k^2-b^2=k^2\cos^2\beta$, and $x^2+y^2=k^2\cos^2\rho$. Reducing, we get

$$\cos^2\rho=\frac{\sin^2\alpha\cos^2\psi+\sin^2\beta\sin^2\psi}{\tan^2\alpha\cos^2\psi+\tan^2\beta\sin^2\psi}. \quad \ldots \quad (29.)$$

Comparing this expression with (13.), we see that

$$\varphi=\psi. \quad \ldots \quad (30.)$$

XI. In the foregoing expressions (17.) and (28.) for the rectification of an arc of a spherical ellipse, the elliptic integrals are of the third order and circular form, with *negative* parameters. We shall now proceed to show that the same arc may be expressed by an elliptic integral of the third order and circular form, having a *positive* parameter.

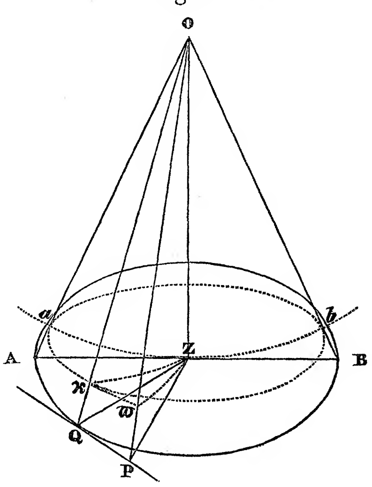
It is shown in most elementary treatises on the integral calculus, in its application to the rectification of plane curves, that if p the perpendicular let fall from a fixed point as pole on a tangent to the curve, makes the angle λ with a fixed right line drawn through the pole, t being the intercept of the tangent between the point of contact and the foot of the perpendicular, we shall have

$$\left.\begin{aligned}\pm s&=\int p\,d\lambda+\frac{dp}{d\lambda}\\ \text{and } t&=-\frac{dp}{d\lambda}\end{aligned}\right\} \quad \ldots \quad (31.)$$

The signs of s to be taken as the curve is concave or convex to the pole.

XII. To investigate an analogous formula for the rectification of a spherical curve, the intersection of a cone of any order with a concentric sphere.

Fig. 3.



Let a point Z be assumed on the surface of the sphere as pole, and through this point a tangent plane ZAQB, or (Θ), to the sphere being drawn, the cone whose vertex is at O, the centre of the sphere, and which passes through the given spherical curve, will cut this tangent plane (Θ) in a plane curve AQB, whose rectification may be effected, when possible, by (31.). Now a tangent plane OQP, or (T), may be conceived as drawn touching the cone, and cutting the tangent plane (Θ) in a right line QP or t, which will be a tangent to the plane curve in (Θ). It will also cut the sphere in an arc of a great circle ($\kappa\varpi$) which will touch the spherical curve in κ. Let the distance QO of the point of contact of the line t with the plane curve from the centre of the sphere be R. Through the centre of the sphere let a plane OZP, or (Π), be drawn at right angles to the straight line t. Now this plane, as it is perpendicular to t, must be perpendicular to the planes (Θ) and (T) which pass through t. As the plane (Π) is perpendicular to the plane (Θ), it must pass through (Z) the point of contact of this plane with the sphere, and cut the plane

of the curve AQB in a right line ZP, or p, which passes through the pole, the point of contact of (Θ) with the sphere. This line p being in (Π) must be perpendicular to t. The plane (Π) will also cut the sphere in an arc of a great circle $Z\varpi = \varpi$, perpendicular to $z\varpi$, the tangent arc to the spherical curve; for these arcs must be at right angles to each other, since the planes in which they lie, (Π) and (T), are at right angles. Let P be the distance OP of the point, in which the plane (Π) cuts the right line t, from the centre of the sphere; r the distance ZQ of the pole of the plane curve to the point in which t touches it, τ being the angle which t subtends at the centre of the sphere, and k its radius,

$$\left.\begin{array}{l} R^2=k^2+r^2,\ P^2=k^2+p^2,\ t^2=r^2-p^2=R^2-P^2 \\ p=k\sin\varpi,\ t=P\tan\tau \end{array}\right\} \quad \text{(32.)}$$

τ is the angle between OQ and OP.

Let ds be the element of an arc of the plane curve between any two consecutive positions of R, indefinitely near to each other; $kd\sigma$ the corresponding element of the spherical curve between the same consecutive positions of R. Then the areas of the elementary triangles on the surface of the cone, between these consecutive positions of R, having their vertices at the centre of the sphere, and for bases the elements of the arcs of the plane and spherical curves respectively, are as their bases multiplied by their altitudes. Let S and S′ be these areas; then

$$S:S'::P\frac{ds}{d\lambda}:k^2\frac{d\sigma}{d\lambda}. \quad \text{(a.)}$$

But the areas of triangles are also as the products of their sides into the sines of the contained angles, *i. e.* in this case as the squares of the sides, or

$$S:S'::R^2:k^2, \quad \text{(b.)}$$

or

$$\frac{d\sigma}{d\lambda}=\frac{P}{R^2}\frac{ds}{d\lambda}; \quad \text{(c.)}$$

putting for ds its value given in (31.),

$$\frac{d\sigma}{d\lambda}=\frac{P}{R^2}\left\{\frac{d^2p}{d\lambda^2}+p\right\}. \quad \text{(d.)}$$

Now $p=P\sin\varpi$, $P^2=R^2-t^2$, and $P^2=k^2+p^2$;

whence
$$P\frac{dP}{d\lambda}=p\frac{dp}{d\lambda}, \text{ and } t=-\frac{dp}{d\lambda}.$$

Substituting these values in (d.),

$$\frac{d\sigma}{d\lambda}=\sin\varpi+\frac{1}{R^2}\left\{P\frac{d^2p}{d\lambda^2}-\frac{dP}{d\lambda}\frac{dp}{d\lambda}\right\}. \quad \text{(e.)}$$

We now proceed to show that the last term of this equation is the differential of the arc, with respect to λ, subtended at the centre of the sphere.

This arc being τ, $\tan\tau=\frac{t}{P}$, $\cos\tau=\frac{P}{R}$.

$$\because \quad \frac{d\tau}{d\lambda}=\frac{1}{R^2}\left\{P\frac{dt}{d\lambda}-t\frac{dP}{d\lambda}\right\}, \quad \ldots \quad \text{(f.)}$$

or as

$$t=-\frac{dp}{d\lambda},\ \frac{d\tau}{d\lambda}=-\frac{1}{R^2}\left\{P\frac{d^2p}{d\lambda^2}-\frac{dp}{d\lambda}\,\frac{dP}{d\lambda}\right\}. \quad \ldots \quad \text{(g.)}$$

Adding this equation to (e.), we get for the final result,

$$\left.\begin{array}{l}\pm\sigma=\int d\lambda\,\sin\varpi-\tau.\\ \text{If } t=\frac{dp}{d\lambda}, \text{ the formula becomes } \pm\sigma=\int d\lambda\,\sin\varpi+\tau.\end{array}\right\} \quad \ldots \quad (33.)$$

Throughout these pages, to avoid circumlocution and needless repetitions, we shall designate as the *pro*-jected tangent, or briefly as the *protangent*, that portion of a tangent to a curve, whether it be a right line, a circle, or a parabola, between its point of contact, and a perpendicular from a fixed point let fall upon it, whether this perpendicular be a right line, or a circular, or a parabolic arc. This definition is the more necessary, as the protangent will continually occur in the following investigations. The term is not inappropriate, as the *pro*-tangent is the *projection* of the radius vector on the tangent.

XIII. To apply the formula (33.) to the rectification of the spherical ellipse.

Let, as before, A and B be the semiaxes of the plane elliptic base of the cone, r the central radius vector drawn to the point of contact of the tangent t, p the perpendicular from the centre on this tangent, t the intercept of the tangent to the plane ellipse between the point of contact and the foot of the perpendicular, λ the angle between p and A. Let α, β, ρ, ϖ, τ be the angles subtended at the centre of the sphere, whose radius is 1, by the lines A, B, r, p, t, we shall consequently have

$$A=k\tan\alpha,\ B=k\tan\beta,\ r=k\tan\rho,\ p=k\tan\varpi, \text{ and } t=\sqrt{k^2+p^2}\tan\tau. \quad \ldots \quad (34.)$$

Now in the plane ellipse

$$p^2=A^2\cos^2\lambda+B^2\sin^2\lambda,$$

therefore in the spherical ellipse

$$\tan^2\varpi=\tan^2\alpha\cos^2\lambda+\tan^2\beta\sin^2\lambda; \quad \ldots \quad (35.)$$

whence

$$\sec^2\varpi=\sec^2\alpha\cos^2\lambda+\sec^2\beta\sin^2\lambda.$$

Dividing the former by the latter,

$$\sin^2\varpi=\frac{\tan^2\alpha\cos^2\lambda+\tan^2\beta\sin^2\lambda}{\sec^2\alpha\cos^2\lambda+\sec^2\beta\sin^2\lambda}. \quad \ldots \quad (36.)$$

Introducing this value of $\sin\varpi$ into (32.), the general form for spherical rectification, the resulting equation will become

$$\sigma=\int d\lambda\left[\frac{\tan^2\alpha\cos^2\lambda+\tan^2\beta\sin^2\lambda}{\sec^2\alpha\cos^2\lambda+\sec^2\beta\sin^2\lambda}\right]^{\frac{1}{2}}-\tau. \quad \ldots \quad (37.)$$

XIV. To reduce this expression to the usual form of an elliptic integral.

Assume

$$\tan\chi=\cos\varepsilon\tan\lambda\ . \quad \ldots \quad (38.)$$

We must first show that this amplitude χ is equal to the amplitude φ in (13.), and therefore to ψ in (25.), as we proved in (X.).

In an ellipse, if ψ and λ are the angles which a central radius vector, and a perpendicular from the centre, on the tangent drawn through its extremity, make with the major axis, we know that $\tan\psi=\frac{B^2}{A^2}\tan\lambda=\frac{\tan^2\beta}{\tan^2\alpha}\tan\lambda$. Introducing this value of $\tan\psi$ into (6.) and reducing,

$$\cos^2\rho=\cos^2\alpha\cos^2\beta\left[\frac{\tan^2\alpha\cos^2\lambda+\tan^2\beta\sin^2\lambda}{\tan^2\alpha\cos^2\beta\cos^2\lambda+\tan^2\beta\cos^2\alpha\sin^2\lambda}\right].$$

Comparing this value of $\cos^2\rho$ with that assumed for $\cos^2\rho$ in (13.), namely,

$$\cos^2\rho=\frac{\sin^2\alpha\cos^2\varphi+\sin^2\beta\sin^2\varphi}{\tan^2\alpha\cos^2\varphi+\tan^2\beta\sin^2\varphi},$$

we get, after some reductions,

$$\tan\varphi=\cos\varepsilon\tan\lambda. \quad \ldots \quad (39.)$$

But in (38.) we assumed $\tan\chi=\cos\varepsilon\tan\lambda$. Hence the amplitudes φ, ψ and χ in (13.), (25.), and (38.) are equal. We may accordingly write φ instead of χ. Substituting the value of $\tan\lambda$, derived from the equation $\tan\varphi=\cos\varepsilon\tan\lambda$, in (38.) the integral in (37.) becomes

$$\int\frac{\cos\alpha\cos\beta[\sin^2\alpha-(\sin^2\alpha-\sin^2\beta)\sin^2\varphi]d\varphi}{[\cos^2\alpha+(\sin^2\alpha-\sin^2\beta)\sin^2\varphi]\sqrt{\sin^2\alpha\cos^2\varphi+\sin^2\beta\sin^2\varphi}}.$$

Now $$\cos\varepsilon=\frac{\cos\alpha}{\cos\beta},\ \tan^2\varepsilon=\frac{\sin^2\alpha-\sin^2\beta}{\cos^2\alpha}\ \sin^2\eta=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}. \quad \ldots \quad (40.)$$

Making the substitutions suggested by these relations and reducing, we get

$$\sigma=\frac{\cos\beta}{\cos\alpha\sin\alpha}\int\left[\frac{d\varphi}{[1+\tan^2\varepsilon\sin^2\varphi]\sqrt{1-\sin^2\eta\sin^2\varphi}}\right]-\frac{\cos\alpha\cos\beta}{\sin\alpha}\int\frac{d\varphi}{\sqrt{1-\sin^2\eta\sin^2\varphi}}-\tau, \quad . \quad (41.)$$

an elliptic integral of the third order, with a *positive* parameter, and therefore of the *circular* form.

This is case IX. in the Table, page 316.

Writing n for $\tan^2\varepsilon$, i for $\sin\eta$, and expressing $\sin\alpha$, $\cos\alpha$, $\sin\beta$, $\cos\beta$ in terms of n and i, (41.) becomes

$$\sigma=\left(\frac{1+n}{n}\right)\sqrt{mn}\int\left[\frac{d\varphi}{[1+n\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}\right]-\frac{i^2}{\sqrt{mn}}\int\frac{d\varphi}{\sqrt{1-i^2\sin^2\varphi}}-\tau. \quad \ldots \quad (42.)$$

XV. To express the *protangent* τ in terms of λ and φ. We found in XII.

$$\tan^2\tau=\frac{t^2}{P^2}=\frac{t^2p^2}{P^2p^2}=\frac{(A^2-B^2)^2\sin^2\lambda\cos^2\lambda}{[k^2+a^2\cos^2\lambda+b^2\sin^2\lambda][a^2\cos^2\lambda+b^2\sin^2\lambda]}.$$

Now $$A=k\tan\alpha,\ B=k\tan\beta,\ e^2=\frac{A^2-B^2}{A^2},\ \text{and}\ \sin^2\varepsilon=\frac{\sin^2\alpha-\sin^2\beta}{\cos^2\beta},$$

whence $$\tan\tau=\frac{e^2\sin\alpha\sin\lambda\cos\lambda}{\sqrt{1-e^2\sin^2\lambda}\sqrt{1-\sin^2\varepsilon\sin^2\lambda}}. \quad \ldots \quad (43.)$$

To express $\tan\tau$ *in terms of the amplitude* ϕ.

Assume the relation established in (13.) or (25.) or (38.) or (39.), $\tan\phi=\cos\varepsilon\tan\lambda$. Introducing this condition into (43.), we obtain

$$\tan\tau=\frac{e\tan\varepsilon\sin\phi\cos\phi}{\sqrt{1-\sin^2\eta\sin^2\phi}}; \quad \ldots \quad (44.)$$

or as $\sqrt{m}=e,\quad \sqrt{n}=\tan\varepsilon,\quad i=\sin\eta,$

the last equation becomes

$$\tan\tau=\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}. \quad \ldots \quad (45.)$$

Hence (42.) may now be written

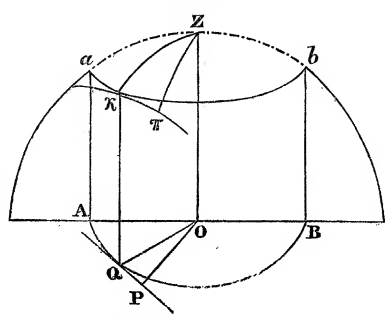
$$\sigma=\left(\frac{1+n}{n}\right)\sqrt{mn}\int\left[\frac{d\phi}{[1+n\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}\right]-\frac{i^2}{\sqrt{mn}}\int\frac{d\phi}{\sqrt{1-i^2\sin^2\phi}}-\tan^{-1}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}\right] \quad (46.)$$

Now this formula and (17.) represent the same arc of the spherical ellipse; they may therefore be equated together. Accordingly

$$\left.\begin{aligned}\left(\frac{1+n}{n}\right)\int\left[\frac{d\phi}{[1+n\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}\right]-\left(\frac{1-m}{m}\right)\int\left[\frac{d\phi}{[1-m\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}\right]\\ =\frac{i^2}{mn}\int\frac{d\phi}{\sqrt{1-i^2\sin^2\phi}}+\frac{1}{\sqrt{mn}}\tan^{-1}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}\right]\end{aligned}\right\} \quad \ldots \quad (47.)$$

This is the well-known theorem established by LEGENDRE, *Traité des Fonctions Elliptiques*, tom. i. p. 68, for the comparison of elliptic integrals of the circular form, with positive and negative parameters respectively. These circular forms arise from treating the element of the spherical conic either as the hypothenuse of an infinitesimal right-angled triangle, or as an element of a circular arc, having the same curvature. When we adopt the former principle, we obtain for the arc an elliptic integral of the third order, circular form and negative parameter. When we choose the latter, we get a circular form of the same order, with a positive parameter. Equating these expressions for the same arc of the curve, the resulting relation is LEGENDRE's theorem. We thus see how an elliptic integral with a *positive* parameter may be made to depend on another with a negative parameter less than 1 and greater than i^2.

Fig. 4.



XVI. We must not confound the angle λ in the preceding article with the angle λ in Art. (X.). Marking the latter λ by a trait thus, $\lambda_{,}$, to distinguish it from the former, we shall investigate the relation between them. Through ZO the axis of the cylinder, let a plane be drawn making the angle ψ with the plane ZOAa. Let this plane cut the spherical ellipse in the point κ, and the plane ellipse the orthogonal projection of the latter in the point Q. Through κ draw an arc of a great circle $\kappa\pi$ touching the curve, and through Q draw a right line touching the plane ellipse. From Z let

fall the perpendicular arc $Z\varpi$ on the tangent arc of the circle, making the angle λ with the arc Za. From O let fall on the tangent to the plane ellipse at Q, the perpendicular OP making the angle $\lambda_{,}$ with OA.

Then $$\tan\lambda=\frac{\tan^2\alpha}{\tan^2\beta}\tan\psi, \text{ and } \tan\lambda_{,}=\frac{\sin^2\alpha}{\sin^2\beta}\tan\psi.$$

Hence we derive $$\frac{\tan\lambda_{,}}{\tan\lambda}=\cos^2\varepsilon. \quad \text{Whence } \tan\lambda.\tan\lambda_{,}=\cos^2\varepsilon\tan^2\lambda.$$

But we have shown in (39.) that

$$\tan^2\phi=\cos^2\varepsilon\tan^2\lambda,$$

whence $$\tan^2\phi=\tan\lambda\tan\lambda_{,}, \quad \ldots\ldots \quad (48.)$$

on *the tangent of the amplitude* ϕ *is a mean proportional between the tangents of the normal angles which a point of contact* $\varkappa$ *on the spherical ellipse and its projection* Q *on the plane ellipse the base of the cylinder produce.*

XVII. We may obtain, under another form, the rectification of the spherical ellipse.

Assume the equations of the right cylinder and generating sphere as given in (19.),

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ and } x^2+y^2+z^2=k^2.$$

Make $$x=a\sin\theta, \quad y=b\cos\theta; \quad \ldots\ldots \quad (49.)$$

hence $$z^2=k^2-a^2\sin^2\theta-b^2\cos^2\theta;$$

and therefore $$k\frac{d\sigma'}{d\theta}=\left[\frac{a^2(k^2-b^2)\cos^2\theta+b^2(k^2-a^2)\sin^2\theta}{(k^2-b^2)\cos^2\theta+(k^2-a^2)\sin^2\theta}\right]^{\frac{1}{2}}. \quad \ldots\ldots \quad (50.)$$

Now

$$a^2(k^2-b^2)=k^4\sin^2\alpha\cos^2\beta, \quad b^2(k^2-a^2)=k^4\sin^2\beta\cos^2\alpha, \quad k^2-b^2=k^2\cos^2\beta, \quad k^2-a^2=k^2\cos^2\alpha.$$

Substituting these values in (50.), and integrating,

$$\sigma'=\int d\theta\left[\frac{\tan^2\alpha\cos^2\theta+\tan^2\beta\sin^2\theta}{\sec^2\alpha\cos^2\theta+\sec^2\beta\sin^2\theta}\right]^{\frac{1}{2}}. \quad \ldots\ldots \quad (51.)$$

If we now compare this formula with (37.) and make $\theta=\lambda$, we shall have

$$\sigma'-\sigma=\tau. \quad \ldots\ldots \quad (52.)$$

Hence we may represent the difference between two arcs of a spherical ellipse, measured from the vertices of the major and minor arcs of the curve, by the arc τ of a great circle which touches the curve.

XVIII. We may thus, by the help of the foregoing theorems, show that when any elliptic integral of the third order and circular form is given, whether the parameter be positive or negative, we may always obtain the elements of the spherical ellipse, of whose arc the given function is the representative.

Let the parameter be negative.

As $$e^2=\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}=m, \quad \text{and } \sin^2\eta=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}=i^2,$$

we shall have $$\tan^2\alpha=\frac{m-i^2}{i^2(1-m)},\quad \tan^2\beta=\frac{m-i^2}{i^2}. \quad \ldots \quad (53.)$$

In order that these values of $\tan\alpha$, $\tan\beta$ may be real, we must have $m>i^2$ and $m<1$.

Let the parameter be positive.

Now $$\tan^2\varepsilon=\frac{\sin^2\alpha-\sin^2\beta}{\cos^2\alpha}=n,\quad \text{and } \sin^2\eta=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}=i^2,$$

hence $$\tan^2\alpha=\frac{n}{i^2},\quad \tan^2\beta=\frac{n}{i^2}\frac{(1-i^2)}{1+n}. \quad \ldots \quad (54.)$$

There is in this case no restriction on the magnitude of n.

XIX. To determine the value of the expression

$$\left(\frac{1+n}{n}\right)\sqrt{mn}\int\left[\frac{d\varphi}{(1+n\sin^2\varphi)\sqrt{1-i^2\sin^2\varphi}}\right],$$

when n is infinite.

As $$m-n+mn=i^2,\text{ or } (1-m)(1+n)=1-i^2=j^2,$$

when n is infinite, $m=1$.

Resuming the expression given in (47.),

$$\sigma=\left(\frac{1+n}{n}\right)\sqrt{mn}\int\left[\frac{d\varphi}{(1+n\sin^2\varphi)\sqrt{1-i^2\sin^2\varphi}}\right]-\frac{i^2}{\sqrt{mn}}\int\frac{d\varphi}{\sqrt{1-i^2\sin^2\varphi}}-\tan^{-1}\left[\frac{\sqrt{mn}\sin\varphi\cos\varphi}{\sqrt{1-i^2\sin^2\varphi}}\right],$$

we find that when n is infinite, α is a right angle.

For $$n=\tan^2\varepsilon=\frac{\sin^2\alpha-\sin^2\beta}{\cos^2\alpha}=\infty,\text{ therefore } \alpha=\frac{\pi}{2}.$$

Now ψ being the angle between the spherical radius vector drawn to the extremity of the arc, and the major principal arc, we have

$$\tan\psi=\frac{\tan^2\beta}{\tan^2\alpha}\tan\lambda,\text{ and } \tan\varphi=\frac{\cos\alpha}{\cos\beta}\tan\lambda,\text{ or } \tan\psi=\frac{\tan\beta}{\tan\alpha}\frac{\sin\beta}{\sin\alpha}\tan\varphi.$$

Hence ψ is indefinitely less than φ, when n is infinite, or when α is a right angle. In this case therefore $\sigma=0$, and we get, when n is infinite, and φ not 0,

$$\left(\frac{1+n}{n}\right)\sqrt{mn}\int\left[\frac{d\varphi}{(1+n\sin^2\varphi)\sqrt{1-i^2\sin^2\varphi}}\right]=\frac{\pi}{2}. \quad \ldots \quad (55.)$$

We might have derived this theorem directly from (47.), by the transformation

$$\sqrt{n}\sin\varphi=\tan\omega.$$

This is case I. in the Table, p. 316.

Section III.—*On the Spherical Parabola.*

XX. It remains now to exhibit a class of spherical conic sections whose rectification may be effected by elliptic integrals of the *first* order.

The curve which is the gnomonic projection of a plane parabola on the surface of a sphere, the focus being the pole, may be rectified by an elliptic integral of the first order.

Let a sphere be described touching the plane of the parabola at its focus. The spherical curve which is the intersection of the sphere with a cone, whose vertex is at its centre, and whose base is the parabola, may be called the *spherical parabola*.

To find the polar equation of this curve.

The polar equation of the parabola, the focus being the pole, is $r=\frac{2g}{1+\cos\omega}$, $4g$ being the parameter of the parabola. Let γ be the angle which g subtends at the centre of the sphere, and ρ the angle subtended by r, then

$$\tan\rho=\frac{2\tan\gamma}{1+\cos\omega}.$$

Let p be the perpendicular from the focus on a tangent to the parabola, μ the angle which this perpendicular makes with the axis of the parabola; $p=\frac{g}{\cos\mu}$. Whence in the spherical curve, as $p=k\tan\varpi$, $g=k\tan\gamma$,

$$\tan\varpi=\frac{\tan\gamma}{\cos\mu}; \qquad (56.)$$

whence

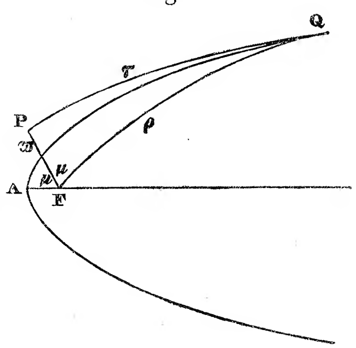
$$\sin\varpi=\frac{\sin\gamma}{\sqrt{1-\cos^2\gamma\sin^2\mu}}. \qquad (57^*.)$$

Introduce this expression into the general form for spherical rectification, $\sigma=\int\sin\varpi d\mu+\tau$, given in (32.), we use the positive sign with τ, since $t=\frac{dp}{d\mu}$.

Now as τ, ϖ and μ are the sides and an angle of a right-angled spherical triangle, since $2\mu=\omega$, we get, by NAPIER's rules, $\tan\tau=\sin\varpi\tan\mu$, whence, by substitution,

$$\sigma=\sin\gamma\int\frac{d\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}+\tan^{-1}\left[\frac{\sin\gamma\tan\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}\right]. \qquad (58.)$$

Fig. 5.



When the sphere becomes indefinitely great the spherical parabola approaches in its contour indefinitely near to the plane parabola. k being the radius of the sphere,

$$\sin\gamma=\tan\gamma=\frac{g}{k},$$

since γ in this case is indefinitely small, whence $\cos^2\gamma=1$. In this manner, since $s=k\sigma$,

* The expression for a perpendicular arc of a great circle let fall from the focus of a spherical ellipse on an arc of a great circle a tangent to this curve, is

$$\sin^2\varpi=\frac{2\sin^2\varepsilon\cos^2\varepsilon\cos^2\mu+(\sin^2\alpha-\sin^2\varepsilon)\cos2\varepsilon\pm\sin\varepsilon\cos\varepsilon\cos\mu\sqrt{\sin^2 2\alpha-\sin^2 2\varepsilon\sin^2\mu}}{(1-\sin^2 2\varepsilon\sin^2\mu)},$$

α being the principal major arc, ε the focal distance, and μ the angle which ϖ makes with α.

When the curve is the spherical parabola, $\alpha+\varepsilon=\frac{\pi}{2}$, $\alpha-\varepsilon=\gamma$, $2\varepsilon=\frac{\pi}{2}-\gamma$, and the preceding expression, when we introduce these relations, will take the very simple form, $\sin\varpi=\frac{\sin\gamma}{\sqrt{1-\cos^2\gamma\sin^2\mu}}$, or $\sin\varpi=1$, as we take the sign $-$ or $+$. See Theory of Elliptic Integrals, p. 31.

2 U 2

(58.) may be transformed into

$$s=g\int\frac{d\mu}{\cos\mu}+g\frac{\sin\mu}{\cos^2\mu},$$

the well-known formula for the rectification of a plane parabola. When, on the other hand, the sphere becomes indefinitely small compared with the parabola, γ approximates to a right angle, and (58.) becomes

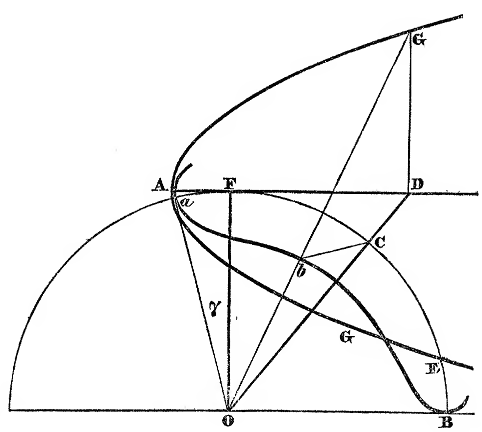
$$s=\mu+\tan^{-1}(\tan\mu)=2\mu,$$

as it should be, since 2μ is the angle which the radius vector ρ makes with the axis.

We shall find the notice of these extreme cases useful.

XXI. Although we have called this curve the spherical parabola, as indicating its mode of generation, it is in fact a closed curve, like all other curves which are the intersections of cones of the second degree with concentric spheres. It is a spherical ellipse, and we shall now proceed to determine its principal arcs.

Fig. 6.



Let ADG be a parabola, F its focus, O being the centre of the sphere which touches the plane of the parabola at F, and being also the vertex of the obtuse-angled cone, of which the parabola ADG is a section parallel to the side of the cone OB. Let the angle AOF or the arc Fa be γ, α and β being the principal semiangles of the cone,

$$2\alpha=\frac{\pi}{2}+\gamma=\text{AOB},$$

whence $$\tan^2\alpha=\frac{1+\sin\gamma}{1-\sin\gamma}.$$

To determine the angle β, or the arc Cb. Bisect the vertical angle AOB of the cone by the line OD, and draw DG an ordinate of the parabola. Then $\tan^2\beta=\left(\frac{\text{DG}}{\text{OD}}\right)^2$. As AOD is an isosceles triangle, $\text{AD}=\text{AO}=\frac{\text{OF}}{\cos\gamma}$; and

$$\text{OD}=\frac{\text{OF}}{\sin\alpha}=\frac{\text{OF}}{\sin\left(\frac{\pi}{4}+\frac{\gamma}{2}\right)}.$$

We have also, as DG is an ordinate of the parabola,

$$\overline{\text{DG}}^2=4\text{AF}\times\text{AD}=4\text{OF}.\tan\gamma\times\frac{\text{OF}}{\cos\gamma}=4\frac{\overline{\text{OF}}^2.\sin\gamma}{\cos^2\gamma}.$$

Hence substituting, $$\tan^2\beta=\frac{2\sin\gamma}{1-\sin\gamma}.$$

We may therefore announce the following important theorem :—

The spherical ellipse, whose principal arcs are given by the equations

$$\tan^2\alpha=\frac{1+\sin\gamma}{1-\sin\gamma},\quad \tan^2\beta=\frac{2\sin\gamma}{1-\sin\gamma}, \qquad (59.)$$

γ being any arbitrary angle, may be rectified by an elliptic function of the first order.

Write x for $\tan\alpha$, y for $\tan\beta$, and eliminate $\sin\gamma$ from the preceding equations,

$$\tan^2\alpha-\tan^2\beta=x^2-y^2=1, \quad \quad (59^*.)$$

the equation of an equilateral hyperbola. We thus obtain the following theorem:—

Any spherical conic section, the tangents of whose principal semiarcs can be the ordinates of an equilateral hyperbola, whose transverse semi-axis is 1, *may be rectified by an elliptic integral of the first order.*

XXII. When we take the complete function, and integrate between the limits 0 and $\frac{\pi}{2}$, we get, not the length of a quadrant of the spherical parabola, as we do when we take the centre as origin, but the length of two quadrants or half the ellipse. We derive also this other remarkable result, that when μ is a right angle, the spherical triangle whose sides are the radius vector, the perpendicular arc on the tangent, and the intercept of the tangent arc between the point of contact and the foot of the perpendicular, is a quadrantal equilateral triangle. For when $\mu=\frac{\pi}{2}$,

$$p=\frac{\pi}{2}, \quad \varpi=\frac{\pi}{2}, \quad \tau=\frac{\pi}{2}.$$

It may also easily be shown, that the arc of a great circle which touches the spherical parabola, intercepted between the perpendicular arcs let fall upon it from the foci, is in every position constant, and equal to a quadrant. See Theory of Elliptic Integrals, p. 35.

Hence the spherical parabola is the envelope of a quadrantal arc of a great circle, which always has its extremities on two fixed great circles of the sphere, the angle between the planes of these circles being $\frac{\pi}{2}+\gamma$.

Resuming the equations given in (59.), which express the tangents of the principal semiarcs of the spherical parabola in terms of $\sin\gamma$, namely,

$$\tan^2\alpha=\frac{1+\sin\gamma}{1-\sin\gamma}, \quad \tan^2\beta=\frac{2\sin\gamma}{1-\sin\gamma},$$

writing i for $\cos\gamma$, and j for $\sin\gamma$, we get

$$\left.\begin{array}{ll} & \tan^2\varepsilon=\frac{1-j}{1+j}, \quad e^2=\frac{1-j}{1+j}\sin^2\eta=\left(\frac{1-j}{1+j}\right)^2, \\ \text{whence} & \tan^2\varepsilon=e^2=\sin\eta=\cos^2\beta. \\ \text{Now} & n=\tan^2\varepsilon, \quad m=e^2; \quad \text{hence } n=m=i. \end{array}\right\} \quad \quad (60.)$$

XXIII. We shall now proceed to the rectification of an arc of the spherical parabola, the centre being the pole. By this method we shall obtain certain geometrical results which have hitherto appeared as mere analytical expressions. In (14.) or (28.) we found for an arc of a spherical ellipse measured from the major principal arc, the following expression, the centre being the pole,

$$\sigma=\frac{\tan\beta}{\tan\alpha}\sin\beta\int\frac{d\psi}{(1-e^2\sin^2\psi)\sqrt{1-\sin^2\eta\sin^2\psi}};$$

or substituting the values of the constants given by the preceding equations,

$$\sigma=\frac{2j}{1+j}\int\frac{d\psi}{\left[1-\left(\frac{1-j}{1+j}\right)\sin^2\psi\right]\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61.)$$

But when the focus is the pole, we found for the arc the following expression in (58.),

$$\sigma=j\int\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}+\tan^{-1}\left[\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}\right].$$

Equating those values of σ, we get the resulting equation,

$$\frac{2j}{1+j}\int\frac{d\psi}{\left[1-\left(\frac{1-j}{1+j}\right)\sin^2\psi\right]\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}=j\int\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}+\tan^{-1}\left[\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}\right] \quad . \quad (62.)$$

XXIV. We shall now show that the amplitudes ψ and μ in the preceding formula are connected by the equation

$$\tan(\psi-\mu)=j\tan\mu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63.)$$

a relation established by LAGRANGE.

Let ϖ and ϖ' be the perpendicular arcs from the centre and focus of the spherical parabola on the tangent arc to the curve. Let λ and μ be the angles which these perpendicular arcs make with the major principal arc. The distance between the centre and focus of the spherical parabola, with the complements of those perpendiculars, constitute the sides of a spherical triangle. We shall therefore have

$$\sin^2\lambda=\sin^2\mu\,\frac{\sec^2\varpi}{\sec^2\varpi'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64.)$$

Now $\sec^2\varpi=\sec^2\alpha\cos^2\lambda+\sec^2\beta\sin^2\lambda$, as in (35.); or writing for $\sec\alpha, \sec\beta$ their particular values in the spherical parabola, given in (59.),

$$\sec^2\varpi=\frac{2}{1-\sin\gamma}-\sin^2\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65.)$$

Again, as $\tan\varpi'=\dfrac{\tan\gamma}{\cos\mu}$,

$$\sec^2\varpi'=\frac{\tan^2\gamma+\cos^2\mu}{\cos^2\mu};$$

reducing (64.), the result is

$$\tan^2\lambda=\frac{2(1+\sin\gamma)}{(\cot\mu-\sin\gamma\tan\mu)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66.)$$

In the case of the spherical parabola,

$$\cos^2\varepsilon=\frac{1+\sin\gamma}{2},$$ whence (66.) becomes

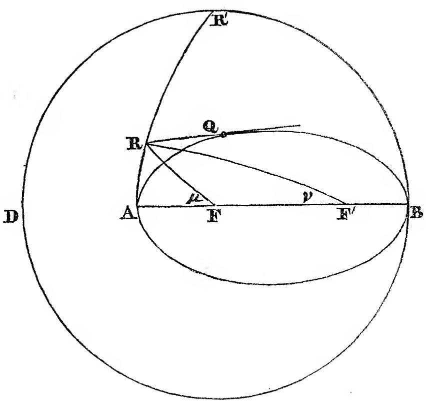
$$\cos\varepsilon\tan\lambda=\frac{1+\sin\gamma}{\cot\mu-\sin\gamma\tan\mu}, \text{ or } \cos\varepsilon\tan\lambda=\frac{\tan\mu+\sin\gamma\tan\mu}{1-\sin\gamma\tan\mu.\tan\mu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67.)$$

The second member of this equation is manifestly the expression for the tangent of the sum of two arcs μ and ν, if we make $\tan\nu=\sin\gamma\tan\mu$.

Hence $$\cos\varepsilon\tan\lambda=\tan(\mu+\nu).$$
In (25.), or (38.) or (39.), we assumed $\tan\psi=\cos\varepsilon\tan\lambda$.
Hence $$\psi=\mu+\nu, \text{ or } \tan(\psi-\mu)=\tan\nu=\sin\gamma\tan\mu.$$

A geometrical interpretation of LAGRANGE's theorem $\tan(\psi-\mu)=\sin\gamma\tan\mu$ may be given by the aid of the spherical parabola.

Fig. 7.



Let $DR_{\prime}B$ be the great circle, the base of the hemisphere, whose pole is F. Let BQA be a spherical parabola, touching the great circle at B, and having one of its foci at F the pole of the hemisphere whose base is the circle $DR_{\prime}B$. Let RQ be an arc of a great circle, a tangent to the curve at Q. From F let fall upon it the perpendicular arc FR. The point R is in the great circle AR which touches the curve at its vertex A. The pole of this circle is the second focus $F_{\prime}$; for $AF_{\prime}=FB=\frac{\pi}{2}$. Let the arcs RF, $RF_{\prime}$ make the angles μ and ν with the transverse arc AB. Hence $AR=\nu$. In the spherical triangle FAR, right-angled at A, we have $\sin AF=\tan\nu\cot\mu$. Now as $AF=\gamma$, $\sin AF=\sin\gamma=j$; and if $\varphi=\mu+\nu$, $\nu=\varphi-\mu$, or reducing, $\tan(\varphi-\mu)=j\tan\mu$; whence we infer that while the original amplitude is the angle μ at the focus F, the derived amplitude φ is the sum of the angles μ and ν at the foci F and $F_{\prime}$.

When the function is complete, or $\mu=\frac{\pi}{2}$, R will coincide with $R_{\prime}$ the pole of the great circle AB, whence ν is also $=\frac{\pi}{2}$, and as $\varphi=\mu+\nu$, $\varphi=\pi$. This shows, that when the function is complete, or the amplitude is a right angle, the amplitude of the derived function will be two right angles.

When the spherical parabola approximates to a great circle of the sphere, the second focus $F_{\prime}$ will approach to F the immoveable focus. The arc $RF_{\prime}$ will, therefore, approach to coincidence with the arc RF, or the angle ν will approximate to μ, so that $\varphi=\mu+\nu=2\mu$ nearly.

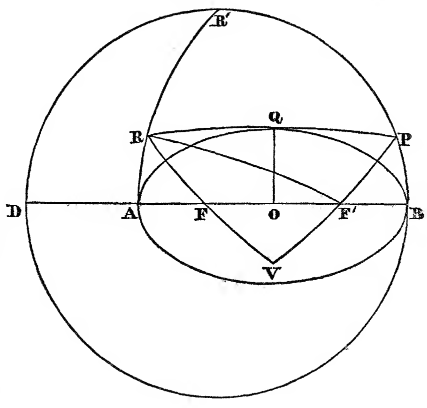
This is the geometrical explanation of the analytical fact observed in this theory, that when the modulus diminishes, or the spherical parabola approximates to a great circle of the sphere, the ratio of any two successive amplitudes approximates to that of two to one.

When the transverse arc of the spherical parabola is a right angle and a half, $\sin\gamma=\frac{1}{\sqrt{2}}$, and if C be its circumference, $C=\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{d\mu}{\sqrt{1-\frac{1}{2}\sin^2\mu}}+\pi$. But two quadrants $2s$, or the loop of a lemniscate, are $=\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{d\mu}{\sqrt{1-\frac{1}{2}\sin^2\mu}}$. Hence $2s=C-\pi$.

Or the loop of a lemniscate is equal to the difference between the circumference of the spherical parabola whose transverse arc is $\frac{3\pi}{2}$, and a semicircle.

When a quadrant of the spherical parabola is taken, or when the point of contact Q coincides with the extremity of the principal minor arc of the curve, we shall have $\phi=\frac{\pi}{2}$.

Fig. 8.



Since in this case RQ=PQ, $FV=F_{\prime}V$, therefore $\mu=OFV=OF_{\prime}V$, or $RF_{\prime}V=\mu+\nu$. As V is the pole of RP, and $F_{\prime}$ is the pole of AR, the point R is the pole of $VF_{\prime}$. Hence $RF_{\prime}V$ is a right angle, but $\mu+\nu=RF_{\prime}V$, whence $\phi=\frac{\pi}{2}$. As $\tan(\phi-\mu)=j\ \tan\mu$, when $\phi=\frac{\pi}{2}$, $\tan\mu=\frac{1}{\sqrt{j}}$. If in the expression $\tan\tau=\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}$ given in (58.), we substitute this value of $\tan\mu$, we shall get $\tan\tau=1$, or $\tau=\frac{\pi}{4}$.

Hence as two quadrants of a spherical parabola are together double of one, we shall have, writing the integral $\int\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}$ in the abbreviated form $\int\frac{d\mu}{\sqrt{I}}$,

$$j\int_0^{\frac{\pi}{2}}\frac{d\mu}{\sqrt{I}}+\frac{\pi}{2}=2j\int_0^{\tan^{-1}\left(\frac{1}{j}\right)^{\frac{1}{2}}}\frac{d\mu}{\sqrt{I}}+2.\frac{\pi}{4}, \quad \text{or} \int_0^{\frac{\pi}{2}}\frac{d\mu}{\sqrt{I}}=2\int_0^{\tan^{-1}\left(\frac{1}{j}\right)^{\frac{1}{2}}}\frac{d\mu}{\sqrt{I}}. \quad . \quad . \quad (68.)$$

Now when i is nearly 1, $\int\frac{d\mu}{\sqrt{I}}=\log\left(\frac{1+\sin\mu}{\cos\mu}\right)$. Taking this expression between the limits $\mu=0$, and $\mu=\tan^{-1}\left(\frac{1}{j}\right)^{\frac{1}{2}}$, we shall have, since $\sin\mu=\frac{1}{\sqrt{1+j}}$, $\cos\mu=\frac{\sqrt{j}}{\sqrt{1+j}}$, and neglecting j and its powers when added to 1, j being very small,

$$\frac{1+\sin\mu}{\cos\mu}=\frac{2}{\sqrt{j}}, \quad \text{whence} \int_0^{\tan^{-1}\left(\frac{1}{j}\right)^{\frac{1}{2}}}\frac{d\mu}{\sqrt{I}}=\log\left(\frac{2}{\sqrt{j}}\right).$$

Therefore (68.) gives $$\int_0^{\frac{\pi}{2}}\frac{d\mu}{\sqrt{I}}=\log\left(\frac{4}{j}\right)^* \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69.)$$

XXV. To show that

$$\int\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}=\frac{1}{1+j}\int\frac{d\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}},$$

the amplitudes ψ and μ being connected as before, by the equation $\tan(\psi-\mu)=j\ \tan\mu$.

Since $$\tan\psi=\frac{1+\sin\gamma}{\cot\mu-\sin\gamma\tan\mu}=\frac{1+j}{\cot\mu-j\tan\mu}.$$

* . . . . . . "résultat fort remarquable, déjà signalé par LEGENDRE; mais nous ignorons comment il y est parvenu."—VERHULST, *Traité Élémentaire des Fonctions Elliptiques*, p. 158.

Differentiating this expression with respect to ψ and μ,

$$\frac{(1+j)}{\sin^2\psi}\frac{d\psi}{d\mu}=\frac{\cos^2\mu+j\sin^2\mu}{\cos^2\mu\sin^2\mu}. \qquad (70.)$$

We have also $$\tan^2\psi=\frac{(1+j)^2\sin^2\mu\cos^2\mu}{(\cos^2\mu-j\sin^2\mu)^2}. \qquad (71.)$$

Whence, after some reductions, $$\sin^2\psi=\frac{(1+j)^2\sin^2\mu\cos^2\mu}{1-i^2\sin^2\mu}. \qquad (72.)$$

Multiplying this expression by $\left(\frac{1-j}{1+j}\right)^2$, and reducing,

$$\frac{1}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}=\frac{\sqrt{1-i^2\sin^2\mu}}{\cos^2\mu+j\sin^2\mu}, \qquad (73.)$$

Multiplying together the left-hand members of the equations (70.), (72.) and (73.), and also the right-hand members together, we get, after some obvious reductions, and integrating,

$$\int\frac{d\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}=(1+j)\int\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}. \qquad (74.)$$

This is the well-known relation between two elliptic integrals of the first order whose moduli are i and $\frac{1-j}{1+j}$, or in the common notation, whose moduli are c and $\frac{1-b}{1+b}$.

XXVI. Let τ be the arc whose tangent is $\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}$,

then $$\tan 2\tau=\frac{2j\sin\mu\cos\mu\sqrt{1-i^2\sin^2\mu}}{\cos^4\mu-i^2\sin^4\mu}; \qquad (75.)$$

and combining (71.) and (73.), we shall find

$$\frac{\tan\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}=\frac{(1+j)\sin\mu\cos\mu\sqrt{1-i^2\sin^2\mu}}{\cos^4\mu-j^2\sin^4\mu}. \qquad (76.)$$

Dividing (75.) by (76.), the result becomes $$\tan 2\tau=\frac{\frac{2j}{1+j}\tan\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}. \qquad (77.)$$

We are thus enabled to express τ, the portion of the tangent arc between the point of contact and the foot of the perpendicular arc on it, in terms of ψ instead of μ.

If we introduce this value of τ into (62.) and combine with it the relations established in (74.), the resulting equation will become

$$\left.\begin{aligned}2\int\frac{d\psi}{\left[1-\left(\frac{1-j}{1+j}\right)\sin^2\psi\right]\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}=\int\frac{d\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}\\ +\left(\frac{1+j}{2j}\right)\tan^{-1}\left[\frac{\frac{2j}{1+j}\tan\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}\right]\end{aligned}\right\}. \qquad (78.)$$

Adopting for the moment the ordinary notation of elliptic integrals,

$$m=-c=\frac{1-j}{1+j}, \quad \text{whence } 1+c=\frac{2j}{1+j}.$$

Introducing this notation, the last formula will become

$$2\Pi_c(-c,\psi)=\mathrm{F}_c(\psi)+\frac{1}{1+c}\tan^{-1}\left[\frac{(1+c)\tan\psi}{\sqrt{1-i^2\sin^2\psi}}\right]. \quad \ldots \quad (79.)$$

In the Traité des Fonctions Elliptiques, tom. i. p. 68, we meet with the formula

$$\Pi_c(n,\psi)+\Pi_c\left(\frac{c^2}{n},\psi\right)=\mathrm{F}_c(\psi)+\frac{1}{\sqrt{\alpha}}\tan^{-1}\left[\frac{\sqrt{\alpha}\tan\psi}{\sqrt{1-c^2\sin^2\psi}}\right]. \quad \ldots \quad (80.)$$

Now when $n=-c$, this formula becomes

$$2\Pi_c(-c,\psi)=\mathrm{F}_c(\psi)+\frac{1}{1+c}\tan^{-1}\left[\frac{(1+c)\tan\psi}{\sqrt{1-c^2\sin^2\psi}}\right], \quad \ldots \quad (81.)$$

whence (79.) and (80.) are identical.

XXVII. Let us now proceed to rectify the spherical parabola by the formula for rectification given in (47.), the centre being the pole. For this purpose, resuming the formula for rectification established in (41.), and deducing the values of the parameter, modulus and coefficients in that expression from the given relations,

$$\tan^2\alpha=\frac{1+\sin\gamma}{1-\sin\gamma}=\frac{1+j}{1-j}, \quad \tan^2\beta=\frac{2\sin\gamma}{1-\sin\gamma}=\frac{2j}{1-j}, \quad \ldots \quad (82.)$$

we get

$$\left.\begin{array}{l}
\text{The parameter, } \tan^2\varepsilon=\dfrac{1-j}{1+j} \\[2ex]
\text{The modulus, } \quad \sin\eta=\dfrac{1-j}{1+j} \\[2ex]
\text{The coefficient } \dfrac{\cos\beta}{\sin\alpha\cos\alpha}=\dfrac{2}{1+j}, \quad \text{the coefficient } \dfrac{\cos\alpha\cos\beta}{\sin\alpha}=\dfrac{1-j}{1+j} \\[2ex]
\text{and} \qquad e\tan\varepsilon=\dfrac{1-j}{1+j}
\end{array}\right\} \quad \ldots \quad (83.)$$

Making these substitutions in (41.), the resulting equation will become

$$\left.\begin{array}{l}
\sigma=\dfrac{2}{(1+j)}\displaystyle\int\frac{d\psi}{\left[1+\left(\frac{1-j}{1+j}\right)\sin^2\psi\right]\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}} \\[3ex]
\quad -\dfrac{(1-j)}{(1+j)}\displaystyle\int\frac{d\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}-\tan^{-1}\left[\frac{\left(\frac{1-j}{1+j}\right)\sin\psi\cos\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}\right]
\end{array}\right\} \quad \ldots \quad (84.)$$

But from (58.), the focus being the pole, we derive

$$\sigma=j\int\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}+\tan^{-1}\left[\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}\right]. \quad \ldots \quad (85.)$$

In (74.) we showed that

$$\int\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}=\frac{1}{1+j}\int\frac{d\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}.$$

Introducing this relation into the last formula, and equating together the equivalent expressions for the arcs in (84.) and (85.), we get for the resulting equation,

$$\left.\begin{aligned} 2\int\frac{d\psi}{\left[1+\left(\frac{1-j}{1+j}\right)\sin^2\psi\right]\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}} &= \int\frac{d\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}} \\ +(1+j)\tan^{-1}\left[\frac{\left(\frac{1-j}{1+j}\right)\sin\psi\cos\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}\right] &+ (1+j)\tan^{-1}\left[\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}\right] \end{aligned}\right\} \quad (86.)$$

We shall now proceed to show that the common formula for the comparison of elliptic integrals having the same modulus and amplitude but reciprocal parameters, is, in this particular case, identical with the geometrical theorem just established.

The formula is, in the ordinary notation,

$$2\Pi_c(c,\psi)=F_c(\psi)+\frac{1}{1+c}\tan^{-1}\left[\frac{(1+c)\tan\psi}{\sqrt{1-c^2\sin^2\psi}}\right]. \quad (87.)$$

We must accordingly show that, c being $\tan^2\varepsilon$, and therefore $\frac{1}{1+c}=\frac{1+j}{2}$

$$\left.\begin{aligned} (1+j)\tan^{-1}\left[\frac{\left(\frac{1-j}{1+j}\right)\sin\psi\cos\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}\right] &+ (1+j)\tan^{-1}\left[\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}\right] \\ =\frac{(1+j)}{2}\tan^{-1}\left[\frac{(1+\tan^2\varepsilon)\tan\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}\right] & \end{aligned}\right\} \quad (88.)$$

If we write τ, τ' and Θ for these angles respectively, we have to show that

$$\Theta=2(\tau+\tau'). \quad (89.)$$

$\tau+\tau'$ is the arc of the great circle, which touches the spherical parabola, intercepted between the perpendicular arcs let fall from the centre and focus upon it.

We must, in the first place, by the help of LAGRANGE's equation between the amplitudes, established on geometrical principles in XXIV., reduce these angles to a single variable. μ is taken as the independent variable instead of ψ, as the trigonometrical function of ψ in terms of μ is in the first power only.

We have, therefore,

$$\left.\begin{aligned} \tan\Theta &= \frac{2\tan\psi}{(1+j)\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}} \\ \tan\tau &= \frac{\left(\frac{1-j}{1+j}\right)\sin\psi\cos\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}} \\ \tan\tau' &= \frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}} \end{aligned}\right\} \quad (90.)$$

The equation between the amplitudes ψ and μ,

$$\tan(\psi-\mu)=j\tan\mu, \text{ gives}$$

$$\tan\psi=\frac{(1+j)\sin\mu\cos\mu}{\cos^2\mu-j\sin^2\mu}. \qquad (91.)$$

Eliminating ψ by the help of this equation, from the value of $\tan\tau$ given in the preceding group,

$$\tan\tau=\frac{(1-j)\sin\mu\cos\mu}{\sqrt{1-i^2\sin^2\mu}}\times\frac{\cos^2\mu+j\sin^2\mu}{\cos^2\mu-j\sin^2\mu}.$$

Using this transformation and reducing,

$$\tan(\tau+\tau')=\tan\mu\sqrt{1-i^2\sin^2\mu}, \qquad (92.)$$

a simple expression for the length of the tangent arc to the spherical parabola between the perpendicular arcs let fall from the centre and focus upon it.

From the last equation we may derive

$$\tan 2(\tau+\tau')=\frac{2\sin\mu\cos\mu\sqrt{1-i^2\sin^2\mu}}{\cos^4\mu-j^2\sin^4\mu}. \qquad (93.)$$

Using the preceding transformations, we may show that

$$\tan\Theta=\frac{2\sin\mu\cos\mu\sqrt{1-i^2\sin^2\mu}}{\cos^4\mu-j^2\sin^4\mu}.$$

Hence

$$\Theta=2(\tau+\tau'). \qquad (94.)$$

Therefore (86.) becomes

$$2\int\frac{d\psi}{\left[1+\left(\frac{1-j}{1+j}\right)\sin^2\psi\right]\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}-\int\frac{d\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}=(1+j)\frac{\Theta}{2}=(1+j)(\tau+\tau'). \qquad (95.)$$

We have thus shown that in the particular case of the general formula for comparing elliptic functions of the third order with reciprocal parameters, when the parameter is *positive* and equal to the *modulus*, the circular arc in the formula of comparison (87.) is equal to twice the arc of the great circle touching the curve and intercepted between the perpendicular arcs let fall from the centre and focus upon it.

If we take the parameter with a *negative* sign, the circular arc τ in (62.) will represent the tangent arc between the point of contact and the foot of the focal perpendicular.

The spherical parabola, like any other spherical ellipse, may be considered as the intersection of an elliptic cylinder with a sphere whose centre is on the axis of the cylinder.

Let a and b be the semiaxes of the base of the cylinder, and k the radius of the sphere, α and β being the principal semiarcs of the spherical parabola,

$$\tan^2\alpha=\frac{a^2}{k^2-a^2}, \qquad \tan^2\beta=\frac{b^2}{k^2-b^2};$$

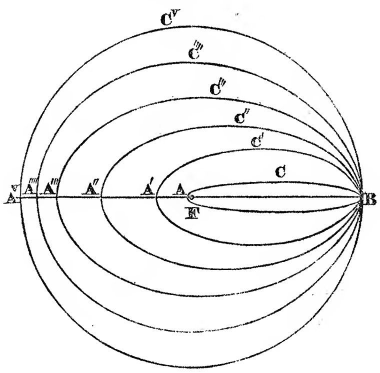
but in (59*.) we found $\tan^2\alpha-\tan^2\beta=1$; hence substituting,

$$k^2=a^2(1+i). \qquad (96.)$$

XXVIII. The foregoing investigations furnish us with the geometrical interpretation of the transformations of LAGRANGE. Let the successive amplitudes ϕ, ψ, χ of the derived functions, be connected by the equations

$$\tan(\phi-\mu)=j\tan\mu, \quad \tan(\psi-\phi)=j_{,}\tan\phi, \quad \tan(\chi-\psi)=j_{,,}\tan\psi. \quad . \quad . \quad . \quad . \qquad (97.)$$

Fig. 9.



We may imagine a series of confocal parabolas having a common axis, described on a plane in contact with a sphere at their common focus. These parabolas will generate a series of confocal spherical parabolas on the surface of the sphere, BCA, BC′A′, BC″A″, BC‴A‴, which will all mutually touch at the vertex B remote from the common focus F. Let the distances between the common focus F and the vertices of the plane parabolas subtend at the centre of the sphere, angles γ, γ', γ'', &c., whose cosines i, $i_{,}$, $i_{,,}$, &c. are connected by the equations

$$i_{,}=\frac{1-\sqrt{1-i^2}}{1+\sqrt{1-i^2}}, \quad i_{,,}=\frac{1-\sqrt{1-i_{,}^2}}{1+\sqrt{1-i_{,}^2}}, \quad i_{,,,}=\frac{1-\sqrt{1-i_{,,}^2}}{1+\sqrt{1-i_{,,}^2}}\ldots \&c., \quad . \quad . \qquad (98.)$$

it is plain that $\gamma=$FA, $\gamma'=$FA′, $\gamma''=$FA″, $\gamma'''=$FA‴, &c.

We may repeat this construction successively, until the parameter of the last of the applied tangent plane parabolas shall become so indefinitely small, compared with the radius of the sphere, that it may ultimately be taken to coincide with its projection. We shall in this way reduce, at least geometrically, the calculation of an elliptic integral of the first order to the rectification of an arc of a parabola, that is, to a logarithm, as in XX. If, on the contrary, the moduli i, $i_{,}$, $i_{,,}$, &c. proceed in a descending series, the angles γ, $\gamma_{,}$, $\gamma_{,,}$ continually increase, the magnitudes of the confocal applied parabolas increase, till at length their parameters become so large, compared with the radius of the sphere, that their central projections pass into great circles of the sphere. The evaluation of the elliptic integral will therefore ultimately be reduced to the rectification of a circular arc. These are the well-known results of the modular transformation of LAGRANGE.

The formulæ established in (58.) for the rectification of the spherical parabola, give

$$\sigma=\sin\gamma\int\frac{d\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}+\tan^{-1}\left[\frac{\sin\gamma\tan\mu}{\sqrt{1-\cos^2\gamma\sin\mu}}\right];$$

or writing i for $\cos\gamma$, j for $\sin\gamma$, and $\sqrt{\mathrm{I}}$ for $\sqrt{1-i^2\sin^2\mu}$,

$$\sigma-\tau=j\int\frac{d\mu}{\sqrt{\mathrm{I}}},$$

σ' and τ' being the corresponding quantities for the next derived spherical parabola,

$$\sigma'-\tau'=j_{,}\int\frac{d\phi}{\sqrt{\mathrm{I}_{,}}}.$$

Now $j_{,}=\frac{2\sqrt{j}}{1+j}$, and $\int\frac{d\mu}{\sqrt{I}}=\frac{1}{1+j}\int\frac{d\phi}{\sqrt{I_{,}}}$, as in (98.) and (74.),

whence $$2(\sigma-\tau)=\sqrt{j}(\sigma'-\tau'), \quad (99.)$$

Thus a simple ratio exists between the arcs, diminished by the protangents, of two consecutive confocal spherical parabolas.

When the functions are complete, μ is taken between 0 and $\frac{\pi}{2}$; ϕ therefore, as in article XXIV., must be taken between 0 and π; but when the amplitude is taken between 0 and π the function is doubled. Moreover, when the functions are complete, the point Q coincides with B; so that in this case the complete function represents, not one, but two quadrants of the spherical parabola, the focus being the pole. Hence as $\tau=\frac{\pi}{2}$, $\tau'=\pi$.

Whence putting C, C′, C″, C‴, &c. for the circumferences of the successive confocal spherical parabolas, derived by the preceding law, we may write

$$\left.\begin{aligned} C-\pi&=\sqrt{j}\,(C_{,}-\pi)\\ C_{,}-\pi&=\sqrt{j_{,}}\,(C_{,,}-\pi)\\ C_{,,}-\pi&=\sqrt{j_{,,}}\,(C_{,,,}-\pi)\\ C_{,,,}-\pi&=\sqrt{j_{,,,}}\,(C_{\text{IV}}-\pi)\\ C_{\text{IV}}-\pi&=\sqrt{j_{\text{IV}}}\,(C_{\text{V}}-\pi) \end{aligned}\right\} \quad (100.)$$

Multiplying successively by the square roots of $j\ j_{,}, j_{,,}, j_{,,,}$, &c., adding and stopping at the fifth derived parabola,

$$C-\pi=\sqrt{j\,j_{,}j_{,,}j_{,,,}j_{\text{IV}}\ \&c.}\ (C_{\text{V}}-\pi).$$

Let this coefficient be $\sqrt{Q}$, and we shall have $C-\pi=\sqrt{Q}(C_{\text{V}}-\pi)$. . . . (101.)

Now we may extend this series, until the last of the derived spherical parabolas shall differ as little as we please from a great circle of the sphere. Let the circumference of this last derived spherical parabola be $C_{,}$. Then $C_{,}=2\pi$, and (101.) becomes

$$C=\pi(1+\sqrt{Q'}). \quad (102.)$$

Hence calculating the quantity Q′, we may express the circumference of a spherical parabola by the circumference of a circle.

When all the spherical parabolas are nearly great circles of the sphere, $i=i_{,}=i_{,,}=i_{,,,}=0$, nearly; and $j=j_{,}=j_{,,}=j_{,,,}=1$, nearly. Whence $Q'=1$, nearly; or

$$C=2\pi \quad (103.)$$

When the spherical parabolas are indefinitely diminished,

$i=i_{,}=i_{,,}=1$, nearly, and $j=j_{,}=j_{,,}=j_{,,,}=0$, therefore $Q'=0$ nearly;

or $$C=\pi. \quad (104.)$$

Hence the circumferences of all spherical parabolas lie between two and four quadrants of a great circle of the sphere.

XXIX. Denoting the angles at the centre of the sphere, subtended by the halves of the semiparameters of the applied confocal parabolas, by γ, γ', γ'', &c., we have $\cos\gamma=i$, $\cos\gamma'=i_{,}$, $\cos\gamma''=i_{,,}$, $\cos\gamma'''=i_{,,,}$, and $\sin\gamma=j$, $\sin\gamma'=j_{,}$, $\sin\gamma''=j_{,,}$, $\sin\gamma'''=j_{,,,}$.

We may, using successively the equation $i_{,}=\frac{1-\sqrt{1-i^2}}{1+\sqrt{1-i^2}}$, determine in terms of j the successive values of i, $i_{,}$, $i_{,,}$, $i_{,,,}$, and of $j_{,}$, $j_{,,}$, $j_{,,,}$, &c., as follows:—

$$\left.\begin{array}{l} i_{,}=\frac{1-j}{1+j},\ i_{,,}=\left[\frac{1-j^{\frac{1}{2}}}{1+j^{\frac{1}{2}}}\right]^2,\ i_{,,,}=\left[\frac{(1+j)^{\frac{1}{2}}-2^{\frac{1}{2}}j^{\frac{1}{4}}}{(1+j)^{\frac{1}{2}}+2^{\frac{1}{2}}j^{\frac{1}{4}}}\right]^2,\ i_{\text{IV}}=\left[\frac{1+j^{\frac{1}{2}}-2^{\frac{1}{2}}2^{\frac{1}{4}}(1+j)^{\frac{1}{4}}j^{\frac{1}{8}}}{1+j^{\frac{1}{2}}+2^{\frac{1}{2}}2^{\frac{1}{4}}(1+j)^{\frac{1}{4}}j^{\frac{1}{8}}}\right]^2 \\ i_{\text{v}}=\left[\frac{(1+j)^{\frac{1}{2}}+2^{\frac{1}{2}}j^{\frac{1}{4}}-2^{\frac{1}{2}}2^{\frac{1}{4}}2^{\frac{1}{8}}(1+j^{\frac{1}{2}})^{\frac{1}{2}}(1+j)^{\frac{1}{8}}j^{\frac{1}{16}}}{(1+j)^{\frac{1}{2}}+2^{\frac{1}{2}}j^{\frac{1}{4}}+2^{\frac{1}{2}}2^{\frac{1}{4}}2^{\frac{1}{8}}(1+j^{\frac{1}{2}})(1+j)^{\frac{1}{8}}j^{\frac{1}{16}}}\right]^2 \text{ \&c.} \end{array}\right\} \quad . \quad (105.)$$

Hence we may derive the successive values of $j_{,}$, $j_{,,}$, $j_{,,,}$ in terms of j.

$$\left.\begin{array}{l} \text{For}\quad j_{,}^2=\frac{2^2 j}{(1+j)^2},\quad j_{,,}^2=\frac{2^2 2\, j^{\frac{1}{2}}(1+j)}{(1+j^{\frac{1}{2}})^4},\quad j_{,,,}^2=\frac{2^2 2^1 2^{\frac{1}{2}} j^{\frac{1}{4}}(1+j)^{\frac{1}{2}}(1+j^{\frac{1}{2}})^2}{[(1+j)^{\frac{1}{2}}+2^{\frac{1}{2}}j^{\frac{1}{4}}]^4}, \\ j_{\text{IV}}^2=\frac{2^2 2^1 2^{\frac{1}{2}} 2^{\frac{1}{4}}(1+j^{\frac{1}{2}})(1+j)^{\frac{1}{4}}j^{\frac{1}{8}}[(1+j)^{\frac{1}{2}}+2^{\frac{1}{2}}j^{\frac{1}{4}}]^2}{[(1+j^{\frac{1}{2}})+2^{\frac{1}{2}}2^{\frac{1}{4}}(1+j)^{\frac{1}{4}}j^{\frac{1}{8}}]^4}, \\ j_{\text{v}}^2=\frac{(2^2 2^1 2^{\frac{1}{2}} 2^{\frac{1}{4}} 2^{\frac{1}{8}})(1+j^{\frac{1}{2}})^{\frac{1}{2}}(1+j)^{\frac{1}{8}}j^{\frac{1}{16}}[(1+j)^{\frac{1}{2}}+2^{\frac{1}{2}}j^{\frac{1}{4}}][(1+j^{\frac{1}{2}})+2^{\frac{1}{2}}2^{\frac{1}{4}}(1+j)^{\frac{1}{4}}j^{\frac{1}{8}}]^2}{[(1+j)^{\frac{1}{2}}+2^{\frac{1}{2}}j^{\frac{1}{4}}+2^{\frac{1}{2}}2^{\frac{1}{4}}2^{\frac{1}{8}}(1+j^{\frac{1}{2}})^{\frac{1}{2}}(1+j)^{\frac{1}{8}}j^{\frac{1}{16}}]^4} \end{array}\right\} \quad . \quad (106.)$$

We may express the coefficient Q, or the continued product of j, $j_{,}$, $j_{,,}$, $j_{,,,}$, &c., in terms of j, the complement of the original modulus. Including in our approximation the fifth derived modulus, we get

$$Q=\frac{(2)^1.(2)^{1+\frac{1}{2}}.(2)^{1+\frac{1}{2}+\frac{1}{4}}.(2)^{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}.(2)^{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}}(jj^{\frac{1}{2}}j^{\frac{1}{4}}j^{\frac{1}{8}}j^{\frac{1}{16}})}{(1+j)^{\frac{1}{16}}(1+j^{\frac{1}{2}})^{\frac{1}{4}}[(1+j)^{\frac{1}{2}}+2^{\frac{1}{2}}j^{\frac{1}{4}}]^{\frac{1}{2}}[(1+j^{\frac{1}{2}})+2^{\frac{1}{2}}2^{\frac{1}{4}}(1+j)^{\frac{1}{4}}j^{\frac{1}{8}}][(1+j)^{\frac{1}{2}}+2^{\frac{1}{2}}j^{\frac{1}{4}}+2^{\frac{1}{2}}2^{\frac{1}{4}}2^{\frac{1}{8}}(1+j^{\frac{1}{2}})^{\frac{1}{2}}(1+j)^{\frac{1}{8}}j^{\frac{1}{16}}]^2} \quad (107.)$$

XXX. It may not be out of place here to show, although the investigation more properly belongs to another part of the subject, that the arc of a spherical parabola may be represented as the sum of two elliptic integrals of the third order, having imaginary parameters; or in other words, that every elliptic integral of the *first* order may be exhibited as the sum of two elliptic integrals of the third order, having *imaginary reciprocal* parameters.

Assume the expression given in (58.) for an arc of the spherical parabola, the focus being the pole, and μ the angle which the perpendicular arc from the focus, on the tangent arc of a great circle to the curve, makes with the principal transverse arc,

$$\sigma=\sin\gamma\int\frac{d\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}+\tan^{-1}\left\{\frac{\sin\gamma\tan\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}\right\}.$$

Let $\cos\gamma=i$, $\sin\gamma=j$, and to preserve uniformity in the notation, write ϕ for μ. Then differentiating the preceding equation, it becomes after some reductions,

$$\frac{d\sigma}{d\phi}=\frac{j[1-i^2\sin^2\phi+\cos^2\phi+j^2\sin^2\phi]}{[\cos^2\phi-i^2\sin^2\phi\cos^2\phi+j^2\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}. \quad . \quad . \quad . \quad (a.)$$

Now the numerator is equivalent to $2j(1-i^2\sin^2\phi)$, and the denominator may be written in the form $1-2i^2\sin^2\phi+i^2\sin^4\phi$. But $i^2=i^2(i^2+j^2)$, hence this last expression may be put under the form $1-2i^2\sin^2\phi+i^4\sin^4\phi+i^2j^2\sin^4\phi$. This expression is the sum of two squares. Resolving this sum into its constituent factors, we get

$$\frac{d\sigma}{d\phi}=\frac{2j(1-i^2\sin^2\phi)}{[1-i(i+j\sqrt{-1})\sin^2\phi][1-i(i-j\sqrt{-1})\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}. \quad . \quad . \quad (b.)$$

Now this product may be resolved into the sum of two terms. Let

$$\frac{d\sigma}{d\phi}=\frac{\mathrm{P}}{[1-i(i+j\sqrt{-1})\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}+\frac{\mathrm{Q}}{[1-i(i-j\sqrt{-1})\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}. \quad \text{(c.)}$$

Or reducing these expressions to a common denominator,

$$\frac{d\sigma}{d\phi}=\frac{(\mathrm{P}+\mathrm{Q})-(\mathrm{P}+\mathrm{Q})i^2\sin^2\phi+\sqrt{-1}(\mathrm{P}-\mathrm{Q})ij\sin^2\phi}{[1-i(i+j\sqrt{-1})\sin^2\phi][1-i(i-j\sqrt{-1})\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}. \quad \text{(d.)}$$

Hence $$\mathrm{P}+\mathrm{Q}=2j,\ \mathrm{P}-\mathrm{Q}=0;\ \because \mathrm{P}=j,\ \mathrm{Q}=j. \quad \text{(e.)}$$

Integrating (c.), we get

$$\sigma=j\int\frac{d\phi}{[1-i(i+j\sqrt{-1})\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}+j\int\frac{d\phi}{[1-i(i-j\sqrt{-1})\sin^2\phi]\sqrt{1-i^2\sin^2\phi}} \quad (108.)$$

Now if we multiply together the imaginary parameters

$$(i^2+ij\sqrt{-1}) \text{ and } (i^2-ij\sqrt{-1}),$$

their product is i^2, or the parameters are reciprocal.

Since the parameters are each affected with a negative sign, and one is equal to i^2+ a certain quantity, while the other is equal to i^2- a certain quantity, the former parameter is of the *circular* form, while the other is of the *logarithmic* form.

It is very remarkable, that although the spherical parabola is a spherical conic, the imaginary parameters satisfy the criterion of conjugation which belongs to the logarithmic form, and not that which belongs to the circular form. Let $m=i(i-j\sqrt{-1})$, $n=i(i+j\sqrt{-1})$. These values of m and n satisfy the equation of logarithmic conjugation, $m+n-mn=i^2$, and not $m-n+mn=i^2$, the equation of circular conjugation.

On Spherical Conic Sections with Reciprocal Parameters.

XXXI. Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be the equation of an ellipse, the base of an elliptic cylinder. Let two spheres be described, having their centres at the centre of this elliptic base, and intersecting the cylinder in two spherical conic sections. These sections will have reciprocal parameters, if k, k', the radii of the spheres, are connected by the equation

$$(k^2-a^2)(k'^2-a^2)=a^4i^2, \quad (109.)$$

i^2 being, as before, equal to $\frac{a^2-b^2}{a^2}$.

When k and k' are equal, we get $k^2=a^2(1+i)$. This value of k agrees with that found for k in (96.), or, in other words, when the two spheres coincide, the section of the elliptic cylinder by the sphere is a spherical parabola. Hence also the spherical parabola always lies between two spherical conic sections with reciprocal parameters.

Let e^2 and e'^2 be the parameters of those sections of the cylinder made by the spheres. Then, as shown in (12.),

$$e^2=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha\cos^2\beta}=\frac{(a^2-b^2)k^2}{a^2(k^2-b^2)}=\frac{k^2i^2}{k^2-a^2+a^2i^2};$$

but the equation of condition (109.) gives

$$k^2-a^2=\frac{a^4i^2}{k'^2-a^2}, \quad \text{hence } e^2=\frac{k^2(k'^2-a^2)}{a^2k'^2}.$$

In the same manner the spherical conic, whose radius is k', gives

$$e'^2=\frac{k'^2(k^2-a^2)}{a^2k^2}; \quad \therefore\ e^2e'^2=\frac{(k^2-a^2)(k'^2-a^2)}{a^4}=i^2=lm, \quad \ldots \quad (110.)$$

or e^2 and e'^2 are reciprocal parameters.

To compute in this case the value of the coefficient $\frac{\tan\beta}{\tan\alpha}\sin\beta$ in the expression given in (16.) for rectification,

$$\sigma=\frac{\tan\beta}{\tan\alpha}\sin\beta\int\frac{d\varphi}{[1-e^2\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}.$$

Since $$\tan^2\beta=\frac{b^2}{k^2-b^2}, \quad \tan^2\alpha=\frac{a^2}{k^2-a^2}, \quad \sin\beta=\frac{b}{k};$$

we obtain by substitution, $\frac{\tan^2\beta}{\tan^2\alpha}\sin^2\beta=\frac{b^4(k^2-a^2)}{a^2k^2(k^2-a^2+a^2i^2)}$;

but the equation of condition (109.) gives

$$k^2-a^2=\frac{a^4i^2}{k'^2-a^2}, \quad \text{hence } \frac{\tan^2\beta}{\tan^2\alpha}\sin^2\beta=\frac{b^4(k^2-a^2)(k'^2-a^2)}{a^4i^2k^2k'^2}=\frac{b^4}{k^2k'^2}.$$

As this expression is symmetrical, we shall have for the spherical conic section, whose radius is k',

$$\frac{\tan\beta'}{\tan\alpha'}\sin\beta'=\frac{b^2}{kk'}. \quad \ldots \quad (111.)$$

Hence $$\frac{\tan\beta}{\tan\alpha}\sin\beta=\frac{\tan\beta'}{\tan\alpha'}\sin\beta', \quad \ldots \quad (112.)$$

or the coefficients of the elliptic integrals, which determine the arcs of two spherical conic sections, having reciprocal parameters, are equal.

Let $\varkappa$ be the criterion of sphericity; then as

$$\varkappa=(1-m)\left(1-\frac{i^2}{m}\right)=(1-e^2)(1-e'^2)=\frac{b^4}{k^2k'^2},$$

$$\varkappa=\varkappa'. \quad \ldots \quad (113.)$$

XXXII. To determine the values of the angles λ and λ' which correspond to the same angle φ in the expressions for the arcs of spherical conic sections having reciprocal parameters.

Since $$\cos^2\varepsilon=\frac{\cos^2\alpha}{\cos^2\beta}=\frac{k^2-a^2}{k^2-b^2}=\frac{k^2-a^2}{k^2-a^2+a^2i^2}.$$

Introducing the equation of condition $(k^2-a^2)(k'^2-a^2)=a^4i^2$, we get $\cos\varepsilon=\frac{a}{k'}$; but $\tan\varphi=\cos\varepsilon\tan\lambda$, as in (39.); hence $\tan\lambda=\frac{k'}{a}\tan\varphi$, and $\tan\lambda'=\frac{k}{a}\tan\varphi$,

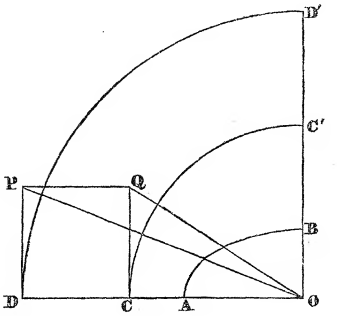
therefore $$k\tan\lambda=k'\tan\lambda', \quad \ldots \quad (114.)$$

or the tangent of the angle λ which the perpendicular arc from the centre of the

spherical conic, on the arc of a great circle touching it, makes with the principal major arc, is inversely as the radius of the sphere.

A simple geometrical construction will give the magnitude of those angles λ and λ'. Let the ellipse OAB be the base of the cylinder; OCC', ODD' being the bases of the hemispheres whose intersections with the cylinders give the spherical conic sections with reciprocal parameters. Erect the equal tangents DP, CQ, and join PO, QO. The angles AOP, AOQ are λ and λ'.

Fig. 10.



When $DP=CQ=0$, $\lambda=\lambda'=0$; when $DP=CQ=\infty$, $\lambda=\lambda'=\frac{\pi}{2}$. The condition (109.) shows that when $k=a$, $k'=\infty$. Now as $k'\tan\lambda'=a\tan\lambda$, is finite always, so long as λ is not absolutely $=\frac{\pi}{2}$; in order that its equal $k'\tan\lambda'$ may be finite also, we must have λ' always equal to 0, for every finite value of $\tan\lambda$.

XXXIII. The tangent of the principal arc of a spherical parabola is a mean proportional between the tangents of the principal arcs of two spherical conics with reciprocal parameters; the three curves being the sections of the same elliptic cylinder by three concentric spheres.

Since $\tan^2\alpha=\frac{a^2}{k^2-a^2}$, $\tan^2\alpha'=\frac{a^2}{k'^2-a^2}$, $\tan^2\alpha\tan\alpha'=\frac{a^4}{(k^2-a^2)(k'^2-a^2)}$.

Introducing the equation of condition $(k^2-a^2)(k'^2-a^2)=a^4i^2$ (109.), we get

$$\tan\alpha\tan\alpha'=\frac{1}{i}. \qquad (115.)$$

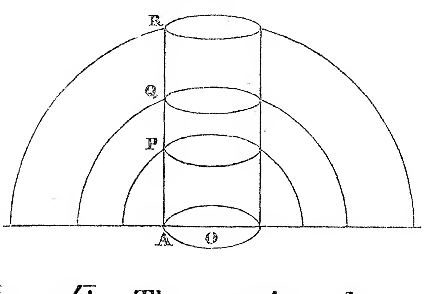
Let k'' be the radius of the sphere whose intersection with the cylinder gives the spherical parabola; then $k''^2=a^2(1+i)$. See (96.)

Hence $k''^2-a^2=a^2i$; and $\tan^2\alpha''=\frac{a^2}{k''^2-a^2}=\frac{1}{i}$; therefore

$$\tan\alpha\tan\alpha'=\tan^2\alpha''. \qquad (116.)$$

The altitudes of the vertices of the three principal major arcs of the two spherical conics with reciprocal parameters, and of the spherical parabola, above the plane of the elliptic base of the cylinder, are in geometrical progression. Let AQ be the altitude of the vertex of the major arc of the spherical parabola. AP, AR the corresponding altitudes of the vertices of the major arcs of the spherical ellipses.

Fig. 11.



Then $AP=\sqrt{k^2-a^2}$, $AR=\sqrt{k'^2-a^2}$, $AQ=\sqrt{k''^2-a^2}=a\sqrt{i}$. The equation of condition gives, as in (109.), $AP\times AR=\overline{AQ}^2$.

We shall give, further on, an expression for the sum of the arcs of two spherical conic sections having the same amplitude, but reciprocal parameters.

XXXIV. The projections of supplemental spherical ellipses on the plane of xy are confocal plane ellipses.

For $\sin\eta=\sin\varepsilon'$, $\sin\eta'=\sin\varepsilon$. Hence $\frac{a^2-b^2}{a^2}=\frac{a_i^2-b_i^2}{k^2-b_i^2}$, $\frac{a_i^2-b_i^2}{a_i^2}=\frac{a^2-b^2}{k^2-b^2}$.

This gives as the resulting value $k^2=a^2+b_i^2=a_i^2+b^2$, or $a^2-b^2=a_i^2-b_i^2$.

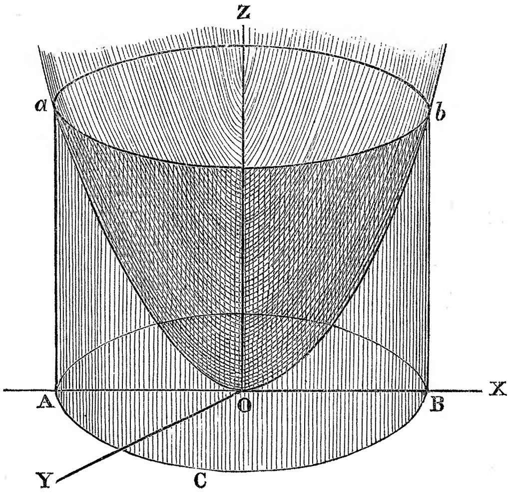
Two supplemental cones are cut by a plane at right angles to their common internal axe. The sections are concentric similar ellipses, having the major and the minor axes of the one, coinciding with the minor and major axes of the other.

For $\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}=e^2$, and $e_i^2=\frac{\tan^2\alpha'-\tan^2\beta'}{\tan^2\alpha'}=\frac{\cot^2\beta-\cot^2\alpha}{\cot^2\beta}=\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}$, or $e'=e$.

SECTION IV.—*On the Logarithmic Ellipse.*

XXXV. The logarithmic ellipse is the curve of symmetrical intersection of a paraboloid of revolution with an elliptic cylinder. This section of the cylinder by the paraboloid is analogous to the section of the cone by the concentric sphere in IX., for this cylinder may be viewed as a cone, having its vertex at the centre of the paraboloid, *i. e.* at an infinite distance.

Fig. 12.



Let the axes of the paraboloid and cylinder coincide with the axis of Z; the vertex of the paraboloid being supposed to touch the plane of xy at the origin O.

Let k be the semiparameter of the paraboloid Oab, and let a and b be the semiaxes of the base of the elliptic cylinder ACB; then the equations of these surfaces, and consequently of the curve in which they intersect, are

$$x^2+y^2=2kz, \text{ and } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \quad . \quad (117.)$$

Let $d\Sigma$ be an element of the required curve, then

$$\frac{d\Sigma}{d\theta}=\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2+\left(\frac{dz}{d\theta}\right)^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (118.)$$

x, y and z being dependent variables on a fourth independent variable θ.

Assume $x=a\cos\theta$, $y=b\sin\theta$, then $a^2\cos^2\theta+b^2\sin^2\theta=2kz$. . . . . . (119.)

Differentiating and substituting,

$$\left(\frac{d\Sigma}{d\theta}\right)^2=a^2\sin^2\theta+b^2\cos^2\theta+\frac{(a^2-b^2)^2}{k^2}\sin^2\theta\cos^2\theta. \quad . \quad . \quad . \quad . \quad . \quad (120.)$$

To reduce this expression to a form suited for integration, it may be written,

$$k^2\left(\frac{d\Sigma}{d\theta}\right)^2=b^2k^2+(a^2-b^2)[k^2+a^2-b^2]\sin^2\theta-(a^2-b^2)^2\sin^4\theta. \quad . \quad . \quad . \quad . \quad (121.)$$

This expression may be reduced as follows:

Let $\quad P=b^2k^2,\quad Q=(a^2-b^2)[k^2+a^2-b^2],\quad R=-(a^2-b^2)^2;$ . . . . . . (122.)

and the preceding equation will become

$$k\Sigma=\int d\theta\sqrt{P+Q\sin^2\theta+R\sin^4\theta}. \quad (123.)$$

Let this trinomial be put under the form of a product of two quadratic factors,

$$(A+B\sin^2\theta)(C-B\sin^2\theta)=AC+B(C-A)\sin^2\theta-B^2\sin^4\theta. \quad (124.)$$

Comparing this expression with the preceding in (121.), we get

$$AC=b^2k^2,\quad C-A=k^2+a^2-b^2,\quad B=a^2-b^2. \quad (125.)$$

To integrate (123.): assume $\qquad \tan^2\varphi=\frac{A+B}{A}\tan^2\theta.$ . . . . (126.)

The limits of integration of the complete functions will continue as before. Making the substitutions indicated by the preceding transformations, the integral will now become

$$\frac{\sqrt{C(A+B)}}{AC}k\Sigma=\int\frac{d\varphi\left[1-\frac{B}{C}\left(\frac{A+C}{A+B}\right)\sin^2\varphi\right]}{\left[1-\frac{B}{A+B}\sin^2\varphi\right]^2\sqrt{1-\frac{B}{C}\left(\frac{A+C}{A+B}\right)\sin^2\varphi}}. \quad (127.)$$

Let $\frac{B}{A+B}=n,\ \frac{B}{C}\frac{(A+C)}{(A+B)}=i^2,\ \frac{A+C}{C}=\frac{i^2}{n},\quad N=1-n\sin^2\varphi,\ I=1-i^2\sin^2\varphi,$ . . (128.)

and the preceding expression may be written

$$\frac{[2n-i^2-n^2]}{\sqrt{n(i^2-n)(1-n)}}\frac{\Sigma}{k}=(1-n)\int\frac{d\varphi I}{N^2\sqrt{I}}; \quad (129.)$$

It will presently be shown that A and C must always have the same sign, whence $i^2>n$.

As $i^2=\frac{1+\frac{A}{C}}{1+\frac{A}{B}}$, and as C is always greater than B, $i^2<1$. From (125.) we may derive

$$\frac{a^2}{k^2}=\frac{(A+B)(C-B)}{(C-A-B)^2},\qquad \frac{b^2}{k^2}=\frac{AC}{(C-A-B)^2}.$$

Now, that the values of a and b may be real, we must have C>B, while A and C must be of the same sign; but as B is essentially positive, C, and therefore A, must be positive.

Since $\qquad \frac{B}{A+B}=n,\quad$ and $\quad\frac{A+C}{C}=\frac{i^2}{n},$ as in (128.)

we may eliminate A, B, C from the values of the semiaxes of the base of the elliptic cylinder, and express a, b and k, in terms of i and n. We may thus obtain

$$\frac{a^2}{k^2}=\frac{n(1-i^2)(i^2-n)}{[2n-i^2-n^2]^2};\qquad \frac{b^2}{k^2}=\frac{n(i^2-n)(1-n)^2}{[2n-i^2-n^2]^2}. \quad (130.)$$

In order that these values of a and b may be real, we must have n positive, $i^2>n$, and $1>i^2$.

This is Case VI. in the Table, p. 316.

If we put c for the eccentricity of the plane elliptic base of the cylinder, we shall have after some obvious reductions, writing f for the complement of c,

$$(1-i^2)(1-c^2)=(1-n)^2, \text{ or } fj=1-n. \qquad (131.)$$

Now this simple equation between n, i and c enables us with great ease to determine the eccentricity c of the base of the elliptic cylinder, whose section with the paraboloid gives the logarithmic ellipse, when we know the parameter n, and the modulus i, of the given elliptic integral.

If we reduce this equation, it becomes $c^2j^2=2n-n^2-i^2$, the denominator of (130.).

XXXVI. To integrate the expression given in (127.), we must assume

$$\Phi_n=\frac{\sin\varphi\cos\varphi\sqrt{1-i^2\sin^2\varphi}}{[1-n\sin^2\varphi]}. \qquad (132.)$$

Differentiate this expression with respect to φ, and we shall have

$$\frac{d\Phi_n}{d\varphi}=\frac{1-2(1+i^2)\sin^2\varphi+3i^2\sin^4\varphi}{[1-n\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}+\frac{2n(\sin^2\varphi-\sin^4\varphi)(1-i^2\sin^2\varphi)}{[1-n\sin^2\varphi]^2\sqrt{1-i^2\sin^2\varphi}}. \qquad (a.)$$

Let $\qquad 1-n\sin^2\varphi=\text{N},\ 1-i^2\sin^2\varphi=\text{I}$, as before.

Separating the numerators of the preceding expression into their component parts, and attaching to each their respective denominators, we shall have

$$\frac{1}{\text{N}\sqrt{\text{I}}}=\frac{1}{\text{N}\sqrt{\text{I}}},\ (b.) \text{ and } -\frac{2(1+i^2)\sin^2\varphi}{\text{N}\sqrt{\text{I}}}=\frac{2(1+i^2)}{n}\frac{(1-n\sin^2\varphi-1)}{\text{N}\sqrt{\text{I}}}=\frac{2(1+i^2)}{n\sqrt{\text{I}}}-\frac{2(1+i^2)}{n\text{N}\sqrt{\text{I}}}. \qquad (c.)$$

The next term gives

$$\frac{3i^2\sin^4\varphi}{\text{N}\sqrt{\text{I}}}=-\frac{3i^2}{n}\frac{(1-n\sin^2\varphi-1)\sin^2\varphi}{\text{N}\sqrt{\text{I}}}=-\frac{3i^2}{n}\frac{\sin^2\varphi}{\sqrt{\text{I}}}+\frac{3i^2\sin^2\varphi}{n\text{N}\sqrt{\text{I}}}. \qquad (d.)$$

Now these two terms may be still further resolved; for

$$-\frac{3i^2}{n}\frac{\sin^2\varphi}{\sqrt{\text{I}}}=\frac{3}{n}\frac{(1-i^2\sin^2\varphi-1)}{\sqrt{\text{I}}}=\frac{3\sqrt{\text{I}}}{n}-\frac{3}{n\sqrt{\text{I}}}, \text{ and}$$

$$\frac{3i^2\sin^2\varphi}{n\text{N}\sqrt{\text{I}}}=-\frac{3i^2}{n^2}\frac{(1-n\sin^2\varphi-1)}{\text{N}\sqrt{\text{I}}}=-\frac{3i^2}{n^2\sqrt{\text{I}}}+\frac{3i^2}{n^2\text{N}\sqrt{\text{I}}},$$

whence (d.) becomes $\qquad \displaystyle\frac{3i^2\sin^4\varphi}{\text{N}\sqrt{\text{I}}}=\frac{3\sqrt{\text{I}}}{n}-\frac{3}{n\sqrt{\text{I}}}-\frac{3i^2}{n^2\sqrt{\text{I}}}+\frac{3i^2}{n^2\text{N}\sqrt{\text{I}}}. \qquad (e.)$

Combining the expressions in (b.), (c.), (d.) or (e.), the first term of the second member of (a.) may be written

$$\frac{[1-2(1+i^2)\sin^2\varphi+3i^2\sin^4\varphi]}{[1-n\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}=\frac{3\sqrt{\text{I}}}{n}+\left[\frac{2}{n}(1+i^2)-\frac{3i^2}{n}-\frac{3}{n}\right]\frac{1}{\sqrt{\text{I}}}+\left[1-\frac{2}{n}(1+i^2)+\frac{3i^2}{n^2}\right]\frac{1}{\text{N}\sqrt{\text{I}}}. \qquad (f.)$$

The second term, $\dfrac{2n(\sin^2\varphi-\sin^4\varphi)\sqrt{\text{I}}}{(1-n\sin^2\varphi)^2}$, of (a.) may be thus developed,

$$\frac{2n\sin^2\varphi\sqrt{\text{I}}}{\text{N}^2}=-\frac{2n}{n}\frac{(1-n\sin^2\varphi-1)\sqrt{\text{I}}}{\text{N}^2}=-\frac{2\text{I}}{\text{N}\sqrt{\text{I}}}+\frac{2\text{I}}{\text{N}^2\sqrt{\text{I}}}; \qquad (g.)$$

and these two latter expressions may be written

$$-\frac{2\text{I}}{\text{N}\sqrt{\text{I}}}=-\frac{2(1-i^2\sin^2\varphi)}{\text{N}\sqrt{\text{I}}}=-\frac{2}{\text{N}\sqrt{\text{I}}}-\frac{2i^2}{n}\frac{(1-n\sin^2\varphi-1)}{\text{N}\sqrt{\text{I}}}=-\frac{2i^2}{n}\frac{1}{\sqrt{\text{I}}}+\frac{2i^2}{n}\frac{1}{\text{N}\sqrt{\text{I}}}-\frac{2}{\text{N}\sqrt{\text{I}}};$$

whence (g.) becomes $$\frac{2n\sin^2\phi\sqrt{\mathrm{I}}}{\mathrm{N}^2}=-\frac{2i^2}{n\sqrt{\mathrm{I}}}-2\left(1-\frac{i^2}{n}\right)\frac{1}{\mathrm{N}\sqrt{\mathrm{I}}}+\frac{2\mathrm{I}}{\mathrm{N}^2\sqrt{\mathrm{I}}}. \quad . \quad . \quad . \quad . \qquad \text{(h.)}$$

The term $-\frac{2n\sin^4\phi\mathrm{I}}{\mathrm{N}^2\sqrt{\mathrm{I}}}$ may be written

$$-\frac{2n\mathrm{I}\sin^4\phi}{\mathrm{N}^2\sqrt{\mathrm{I}}}=-\frac{2\mathrm{I}}{n}\left[\frac{1-2n\sin^2\phi+n^2\sin^4\phi-2+2n\sin^2\phi+1}{\mathrm{N}^2\sqrt{\mathrm{I}}}\right]=-\frac{2\mathrm{I}}{n.\sqrt{\mathrm{I}}}+\frac{4\mathrm{I}}{n.\mathrm{N}\sqrt{\mathrm{I}}}-\frac{2\mathrm{I}}{n.\mathrm{N}^2\sqrt{\mathrm{I}}}. \quad \text{(k.)}$$

Now $$\frac{n\mathrm{I}}{n\sqrt{\mathrm{I}}}=\frac{2}{n}\sqrt{\mathrm{I}},$$

and $$\frac{4\mathrm{I}}{n\mathrm{N}\sqrt{\mathrm{I}}}=\frac{4(1-i^2\sin^2\phi)}{n\mathrm{N}\sqrt{\mathrm{I}}}=\frac{4}{n\mathrm{N}\sqrt{\mathrm{I}}}+\frac{4i^2}{n^2}\,\frac{(1-n\sin^2\phi-1)}{\mathrm{N}\sqrt{\mathrm{I}}},$$

whence $$\frac{4\mathrm{I}}{n\mathrm{N}\sqrt{\mathrm{I}}}=\frac{4i^2}{n^2\sqrt{\mathrm{I}}}-4\frac{(i^2-n)}{n^2}\,\frac{1}{\mathrm{N}\sqrt{\mathrm{I}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(m.)}$$

Combining (k.) with (m.), we shall have

$$-\frac{2n\mathrm{I}\sin^4\phi}{\mathrm{N}^2\sqrt{\mathrm{I}}}=-\frac{2\sqrt{\mathrm{I}}}{n}+\frac{4i^2}{n^2\sqrt{\mathrm{I}}}-\frac{4}{n^2}(i^2-n)\frac{1}{\mathrm{N}\sqrt{\mathrm{I}}}-\frac{2\mathrm{I}}{n\mathrm{N}^2\sqrt{\mathrm{I}}}; \quad . \quad . \quad . \quad . \qquad \text{(n.)}$$

adding (n.) to (h.),

$$\frac{2n(\sin^2\phi-\sin^4\phi)\mathrm{I}}{\mathrm{N}^2\sqrt{\mathrm{I}}}=-\frac{2\sqrt{\mathrm{I}}}{n}+\left(\frac{4i^2}{n^2}-\frac{2i^2}{n}\right)\frac{1}{\sqrt{\mathrm{I}}}+\left[\frac{2i^2}{n}-2+\frac{4}{n}-\frac{4i^2}{n^2}\right]\frac{1}{\mathrm{N}\sqrt{\mathrm{I}}}-2\left(\frac{1}{n}-1\right)\frac{1}{\mathrm{N}^2\sqrt{\mathrm{I}}}; \quad \text{(p.)}$$

adding (f.) and (p.) together, we get as the final result,

$$\frac{d\Phi_n}{d\phi}=\frac{\sqrt{\mathrm{I}}}{n}+\frac{1}{n}\left(\frac{i^2-n}{n}\right)\frac{1}{\sqrt{\mathrm{I}}}+\frac{1}{n^2}[2n-n^2-i^2]\frac{1}{\mathrm{N}\sqrt{\mathrm{I}}}-2\left(\frac{1-n}{n}\right)\frac{\mathrm{I}}{\mathrm{N}^2\sqrt{\mathrm{I}}}; \quad . \quad . \quad . \qquad \text{(q.)}$$

or multiplying by n, transposing and integrating,

$$2(1-n)\int\frac{\mathrm{I}d\phi}{\mathrm{N}^2\sqrt{\mathrm{I}}}=-n\Phi_n+\int d\phi\sqrt{\mathrm{I}}+\left(\frac{i^2-n}{n}\right)\int\frac{d\phi}{\sqrt{\mathrm{I}}}+\left[\frac{2n-n^2-i^2}{n}\right]\int\frac{d\phi}{\mathrm{N}\sqrt{\mathrm{I}}}. \quad . \quad . \qquad \text{(r.)}$$

But we have shown in (129.) that

$$\frac{2[2n-i^2-n^2]}{\sqrt{n(i^2-n)(1-n)}}\,\frac{\Sigma}{k}=2(1-n)\int\frac{\mathrm{I}d\phi}{\mathrm{N}^2\sqrt{\mathrm{I}}},$$

whence $$\frac{2[2n-i^2-n^2]}{\sqrt{n.(i^2-n)(1-n)}}\,\frac{\Sigma}{k}=-n\Phi_n+\int d\phi\sqrt{\mathrm{I}}+\left(\frac{i^2-n}{n}\right)\int\frac{d\phi}{\sqrt{\mathrm{I}}}+\left[\frac{2n-i^2-n^2}{n}\right]\int\frac{d\phi}{\mathrm{N}\sqrt{\mathrm{I}}}. \quad \text{(133.)}$$

Hence, an arc of a logarithmic ellipse may be expressed by a line Φ_n, and in terms of elliptic integrals of the first, second and third orders; the latter being of the logarithmic form (127.) may be written in the form

$$\frac{\Sigma}{k}=\frac{b^2}{\sqrt{\mathrm{C(A+B)}}}\int\frac{d\phi[1-i^2\sin^2\phi]}{[1-n\sin^2\phi]^2\sqrt{1-i^2\sin^2\phi}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(134.)}$$

XXXVII. When the cylinder and the paraboloid are given, we may determine the parameter, modulus and constants of the functions which represent the curve of intersection of these surfaces, in the terms of the constants a, b and k.

The modulus parameter, coefficients and criterion of sphericity may be expressed, as linear products of constants, having simple relations with those of the given surfaces.

Resuming the equations given in (125.),

$$\mathrm{AC}=b^2k^2, \quad \mathrm{C-A}=k^2+a^2-b^2, \quad \mathrm{B}=a^2-b^2,$$

we find $$(A+C)^2=(k^2+a^2-b^2)^2+4b^2k^2.$$

Assume $$4p^2=k^2+(a+b)^2, \quad 4q^2=k^2+(a-b)^2, \quad \ldots \ldots \quad (135.)$$

we shall then have the following equations:—

$$\left.\begin{array}{ll} A+C=4pq, & B=(a+b)(a-b) \\ A+B=(a+p-q)(a+q-p); & C-B=(p+q+a)(p+q-a) \\ A=(b+p-q)(b+q-p); & C=(p+q+b)(p+q-b) \\ ab=(p+q)(p-q), & k^2+a^2+b^2=2(p^2+q^2) \end{array}\right\} \quad \ldots \quad (136.)$$

Substituting these values in (129.) we obtain the resulting expressions

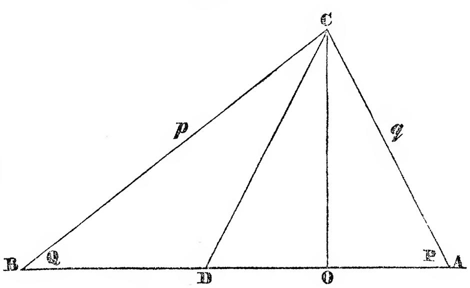
$$\left.\begin{array}{l} i^2=\dfrac{4(a+b)(a-b)pq}{(p+q+b)(p+q-b)(a+p-q)(a+q-p)} \\ n=\dfrac{(a+b)(a-b)}{(a+p-q)(a+q-p)}, \quad m=\dfrac{(a+b)(a-b)}{(p+q+b)(p+q-b)} \end{array}\right\} \quad \ldots \ldots \quad (137.)$$

and if we denote by $\varkappa$ the criterion of sphericity,

$$\varkappa=\frac{-b^4}{a^2(p+q)^2}\left(\frac{p+q+a}{p+q+b}\right)^2\left(\frac{p+q-a}{p+q-b}\right)^2, \quad \ldots \ldots \ldots \quad (138.)$$

we may express the parameters and modulus of the elliptic integral of the third order and logarithmic form by a geometrical construction of remarkable simplicity when the intersecting surfaces are given, or when a, b, and k are given.

Fig. 13.



Take BA$=a$, BD$=b$, and from O the point of bisection of AD, erect the perpendicular OC$=\frac{k}{2}$. Then (135.) gives $p=$BG, $q=$AC, and putting P and Q for the angles BAC and ABC, $a+b=2p\cos Q$, $a-b=2q\cos P$. As p, q, b are the sides of the triangle BCD, and the angle BCD$=$P$-$Q,

$$\cos^2\left(\frac{P-Q}{2}\right)=\frac{(b+p+q)(p+q-b)}{4pq};$$

again as a, p, q are the sides of the triangle ABC, and.

$$\cos^2\left(\frac{P+Q}{2}\right)=\frac{(a+p-q)(a+q-p)}{4pq}.$$

Substituting these values in (137.), we get

$$i^2=\frac{\cos P\cos Q}{\left[\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)\right]^2}, \quad n=\frac{\cos P\cos Q}{\cos^2\left(\frac{P+Q}{2}\right)}, \quad m=\frac{\cos P\cos Q}{\cos^2\left(\frac{P-Q}{2}\right)}, \quad \ldots \quad (139.)$$

and if c be the eccentricity of the elliptic base of the cylinder,

$$c^2=\frac{\sin 2P.\sin 2Q}{\sin^2(P+Q)}. \quad \ldots \ldots \ldots \quad (140.)$$

These are expressions remarkable for their simplicity.

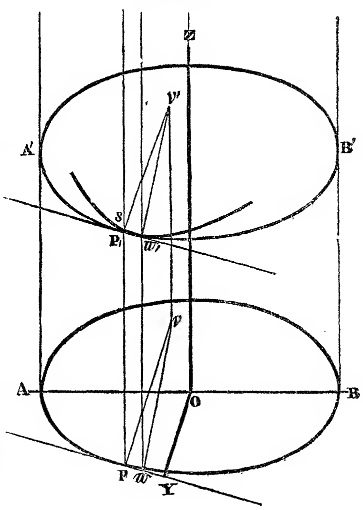
We also find for the criterion of sphericity $\varkappa$,

$$\varkappa=-\left[\frac{\sin^2\left(\frac{P-Q}{2}\right)}{\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)}\right]^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (141.)$$

As $\frac{k}{2}$ is the altitude of a triangle whose sides are a, p, q,

$$a^2k^2=(a+p+q)(p+q-a)(a+q-p)(a+p-q).$$

Fig. 14.



XXXVIII. In the preceding investigations the element of the curve has been taken as a side of a limiting rectilinear polygon inscribed within it. We may however effect the rectification of the curve, starting from other elementary principles. Let APB be the plane base of the elliptic cylinder, and let a series of normal planes $PP'\nu\nu'$ $\varpi\varpi'\nu\nu'$ be drawn to the cylinder, indefinitely near to each other, and parallel to its axis. We may conceive of every element $P\varpi$ of this plane ellipse between the normal planes as the projection of the corresponding element $s\varpi'$ of the logarithmic ellipse. Let τ be the inclination of the element $d\Sigma$ of the logarithmic ellipse to the corresponding element ds of the plane ellipse. We shall have, $d\lambda$ being the elementary angle between the planes $PP'\nu\nu'$ and $\varpi\varpi'\nu\nu'$,

$$\frac{d\Sigma}{d\lambda}=\sec\tau\frac{ds}{d\lambda}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (142.)$$

Now (31.) gives $\frac{ds}{d\lambda}=p+\frac{d^2p}{d\lambda^2}$,

and therefore

$$\Sigma=\int\frac{p}{\cos\tau}d\lambda+\int\frac{d^2p}{d\lambda^2}\sec\tau.d\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (143.)$$

In the plane ellipse $p^2=a^2\cos^2\lambda+b^2\sin^2\lambda$, whence $\frac{d^2p}{d\lambda^2}=-\frac{(a^2-b^2)(a^2\cos^4\lambda-b^2\sin^4\lambda)}{(a^2\cos^2\lambda+b^2\sin^2\lambda)^{\frac{3}{2}}}$. (144.)

We have now to express $\cos\tau$ in terms of λ.

From (119.) combined with (120.) we may derive

$$\sec^2\tau=\frac{d\Sigma^2}{dx^2+dy^2}=\frac{b^2k^2+(a^2-b^2)[k^2+a^2-b^2]\sin^2\theta-(a^2-b^2)^2\sin^4\theta}{k^2(a^2\sin^2\theta+b\cos^2\theta)}. \quad . \quad . \quad . \quad . \quad . \quad (145.)$$

Eliminating $\frac{y}{x}$ between the equations $\tan\lambda=\frac{a^2}{b^2}\frac{y}{x}$, and $\frac{y}{x}=\frac{b}{a}\tan\theta$, we shall have

$$\tan\lambda=\frac{a}{b}\tan\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (146.)$$

If we eliminate $\tan\theta$ by the help of this equation from (145.), we shall obtain

$$\cos^2\tau=\frac{k^2(a^2\cos^2\lambda+b^2\sin^2\lambda)}{a^2k^2+(a^2-b^2)[a^2-b^2-k^2]\sin^2\lambda-(a^2-b^2)^2\sin^4\lambda}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (147.)$$

Substituting this value of $\cos\tau$ in (143.), and writing P′, Q′, R′ for the coefficients of powers of $\sin\lambda$, the resulting equation will become

$$k\Sigma=\int d\lambda\sqrt{P'+Q'\sin^2\lambda+R'\sin^4\lambda}-(a^2-b^2)\int\frac{d\lambda(a^2\cos^4\lambda-b^2\sin^4\lambda)}{k(a^2\cos^2\lambda+b^2\sin^2\lambda)^{\frac{3}{2}}\cos\tau}. \quad . \quad . \quad (148.)$$

As the first of these integrals is precisely similar in form to the integral in (123.), we may in the same manner reduce the expression into factors. Accordingly let

$$P'+Q'\sin^2\lambda+R'\sin^4\lambda=(\alpha+\beta\sin^2\lambda)(\gamma-\beta\sin^2\lambda). \quad . \quad . \quad . \quad . \quad . \quad (149.)$$

Writing α, β, γ instead of A, B, C, and following step by step the investigation in Art. XXXV., we shall have, as in (126.) and (128.), ψ, m, and $i_{\prime}$ being the amplitude, parameter and modulus,

$$\tan^2\psi=\frac{\alpha+\beta}{\alpha}\tan^2\lambda,\quad m=\frac{\beta}{\alpha+\beta},\quad i_{\prime}^2=\frac{\beta}{\gamma}\left(\frac{\alpha+\gamma}{\alpha+\beta}\right). \quad . \quad . \quad . \quad . \quad (150.)$$

As
$$\alpha\gamma=a^2k^2,\quad \beta=a^2-b^2, \text{ and } \gamma-\alpha=a^2-b^2-k^2, \quad . \quad . \quad . \quad . \quad (151.)$$

we shall have the following relations between the constants α, β, γ, m, $i_{\prime}$, and A, B, C, n, i, in (150.) and (128.),

$$\left.\begin{array}{l}\beta=B,\quad \alpha=C-B,\quad \gamma=A+B,\quad \alpha+\gamma=A+C,\\ \gamma-\beta=A,\quad \alpha+\beta=C,\quad \gamma-\alpha-\beta+C-A-B=0,\\ i_{\prime}^2=\frac{\beta(\alpha+\gamma)}{\gamma(\alpha+\beta)}=\frac{B(A+C)}{(A+B)C}=i^2, \text{ or } i_{\prime}=i,\quad m=\frac{\beta}{\alpha+\beta}=\frac{B}{C}.\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (152.)$$

Hence the moduli are the same in the two forms of integration, and the parameters m and n will be found to be connected by the equation $m+n-mn=i^2$; . . (153.) m and n are, therefore, *conjugate* parameters, as they fulfil the condition assumed in (1.).

The amplitudes ϕ and ψ are equal.

In (126.) we assumed, $\tan^2\phi=\frac{A+B}{A}\tan^2\theta$; and in (150.) $\tan^2\psi=\frac{\alpha+\beta}{\alpha}\tan^2\lambda$, but $\tan\lambda=\frac{a}{b}\tan\theta$, as in (146.), whence $\tan^2\psi=\frac{a^2}{b^2}\frac{(\alpha+\beta)A}{(A+B)\alpha}\tan^2\phi$.

In (152.) we have found $\alpha+\beta=C$, and $A+B=\gamma$,

whence $\tan^2\psi=\frac{a^2}{b^2}\frac{AC}{\alpha\gamma}\tan^2\phi$. But $AC=b^2k^2$, and $\alpha\gamma=a^2k^2$, as shown in (125.) and (151.), whence

$$\psi=\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (154.)$$

We shall now proceed to find the value of the second integral in (148.).

From (147.) we may derive $\tan^2\tau=\frac{(a^2-b^2)^2\sin^2\lambda\cos^2\lambda}{k^2(a^2\cos^2\lambda+b^2\sin^2\lambda)}$. . . . . (155.)

Differentiating this expression, reducing, dividing by $\cos\tau$, and integrating, we finally obtain

$$k\int\frac{d\tau}{\cos^3\tau}=(a^2-b^2)\int\frac{d\lambda(a^2\cos^4\lambda-b^2\sin^4\lambda)}{\cos\tau\,(a^2\cos^2\lambda+b^2\sin^2\lambda)^{\frac{3}{2}}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (156.)$$

(148.) may now be written

$$k\Sigma=\int d\lambda\sqrt{P'+Q'\sin^2\lambda+R'\sin^4\lambda}-k^2\int\frac{d\tau}{\cos^3\tau}. \quad \ldots \quad (157.)$$

If we measure the arc of the logarithmic ellipse from the minor principal axe, or from the parabolic arc which is projected into b, instead of placing the origin at the vertex of the major axe as in (119.), we must put

$$x=a\sin\vartheta,\quad y=b\cos\vartheta; \quad \ldots \quad (158.)$$

and following the steps indicated in that article, we shall obtain

$$kS=\int d\vartheta\sqrt{P'+Q'\sin^2\vartheta+R'\sin^4\vartheta}. \quad \ldots \quad (159.)$$

If we now make $\vartheta=\lambda$, and subtract the two latter equations, one from the other, the resulting equation will become

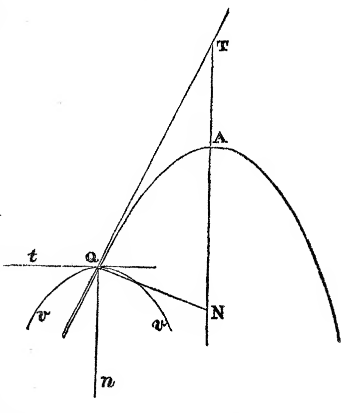
$$S-\Sigma=k\int\frac{d\tau}{\cos^3\tau}. \quad \ldots \quad (160.)$$

But this integral is, we know, the expression for an arc of a common parabola, whose semi-parameter is k, measured from the vertex of the curve, to a point on it, where its tangent makes the angle τ with the ordinate.

Thus the difference between two elliptic arcs measured from the vertices of the curve, which in the plane ellipse may, as we know, be expressed by a right line; and in the spherical ellipse by an arc of a circle, as shown in Art. XV.; will in the logarithmic ellipse be expressed by an arc of a parabola. As a parabolic arc can be rectified only by a logarithm, we may hence see the propriety of the term *logarithmic*, by which this function is designated.

XXXIX. If from the vertex A of a paraboloid, an arc of a parabola be drawn, at right angles to a parabolic section of the paraboloid, it will meet this parabolic section at its vertex. Let the arc AQ be drawn at right angles to the parabolic section Qv of the paraboloid, the point Q is the vertex of the parabola Qv.

Fig. 15.



Draw QT and Qt tangents to the arcs QA and Qv. Then QT and Qt are at right angles. As QT is a tangent to a principal section passing through the axis of the paraboloid, it will meet this axis in a point T; and as Qt is a tangent to the surface of the paraboloid, it will be perpendicular to the normal to the surface QN. Now as Qt is perpendicular to QT and to QN, it is perpendicular to the plane QTN which passes through them, and therefore to every line in this plane, and therefore to the axis AN, or to any line parallel to it, as the diameter Qn. Hence, as the tangent Qt to the parabola Qv is perpendicular to the diameter Qn, Q is the vertex of the parabola.

Hence in the logarithmic ellipse, one extremity of the protangent arc is always the vertex of the parabola which touches the logarithmic ellipse at its other extremity.

This is a very important theorem, as the protangents are arcs of equal parabolas, all measured from the vertices of the parabolas. Hence also the length of the protangent arc depends solely on its normal angle.

As an arc of a circle may be expressed by the notation $s=\sin^{-1}\left(\frac{y}{k}\right)$, y being the ordinate and k the radius, so in like manner an arc of a parabola may be designated by the form $s=\tan^{-1}\left(\frac{y}{k}\right)$; y being the ordinate and k the semiparameter. To distinguish the parabolic arc from the circular arc, the former may be written $s=\tau\alpha\nu^{-1}\left(\frac{y}{k}\right)$. Again, as we say, in the case of the circle, the angle ω and the arc $k\omega$, ω being the angle contained between the normals to the curve at the extremities of the arc: so in the parabola, we may write ω for the angle between the normals, and $(k.\omega)$ for the corresponding parabolic arc. In the case of the parabola the arc is always supposed to be measured from the vertex; in the circle the arc may be measured from any point, as every point is a vertex.

XL. Resuming the equation (157.), $k\Sigma=\int d\lambda\sqrt{P'+Q'\sin^2\lambda+R'\sin^4\lambda}-k^2\int\frac{d\tau}{\cos^3\tau}$. We shall now proceed to develope the first integral of the second side of this equation. As the integral is precisely the same in form as (123.), and the amplitude $\psi=\phi$, as also the modulus $i_{,}=i$, we may substitute α, β, γ for A, B, C, m for n, Φ_m for Φ_n, retaining the modulus and amplitude, which continue unchanged, as we have established in (152.) and (154.); or substituting for α, β, γ their values in m and i, we get

$$\left.\begin{aligned}\frac{2[i^2+m^2-2m]}{\sqrt{m(i^2-m)(1-m)}}\,\frac{\Sigma}{k}=-m\Phi_m-\frac{[i^2+m^2-2m]}{m}\int\frac{d\phi}{[1-m\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}\\+\frac{[i^2-m]}{m}\int\frac{d\phi}{\sqrt{1-i^2\sin^2\phi}}+\int d\phi\sqrt{1-i^2\sin^2\phi}-\frac{[i^2+m^2-2m]\,2}{\sqrt{m(i^2-m)(1-m)}}\int\frac{d\tau}{\cos^3\tau}\end{aligned}\right\}. \quad . \quad (161.)$$

If we eliminate i from the coefficients of (133.) and (161.), putting M for $(1-m\sin^2\phi)$, and N for $(1-n\sin^2\phi)$, as also $\sqrt{\text{I}}$ for $\sqrt{1-i^2\sin^2\phi}$; (133.) will become

$$\frac{2(n-m)}{\sqrt{mn}}\,\frac{\Sigma}{k}=-n\Phi_n+\frac{(1-n)(n-m)}{n}\int\frac{d\phi}{\text{N}\sqrt{\text{I}}}+\frac{m}{n}(1-n)\int\frac{d\phi}{\sqrt{\text{I}}}+\int d\phi\sqrt{\text{I}}, \quad . \quad . \quad (162.)$$

and (161.) will be transformed into

$$\frac{2(n-m)}{\sqrt{mn}}\,\frac{\Sigma}{k}=-m\Phi_m-\frac{(1-m)(n-m)}{m}\int\frac{d\phi}{\text{M}\sqrt{\text{I}}}+\frac{n}{m}(1-m)\int\frac{d\phi}{\sqrt{\text{I}}}+\int d\phi\sqrt{\text{I}}-\frac{2(n-m)}{\sqrt{mn}}\int\frac{d\tau}{\cos^3\tau}. \quad (163.)$$

If we compare together (162.) and (163.), which are expressions for the same arc of the logarithmic ellipse, and make the obvious reductions, putting for Φ_n and Φ_m their values $\frac{\sin\phi\cos\phi\sqrt{\text{I}}}{\text{N}}$ and $\frac{\sin\phi\cos\phi\sqrt{\text{I}}}{\text{M}}$, we shall get the following resulting equation of comparison,

$$\left(\frac{1-n}{n}\right)\int\frac{d\phi}{\text{N}\sqrt{\text{I}}}+\left(\frac{1-m}{m}\right)\int\frac{d\phi}{\text{M}\sqrt{\text{I}}}=\frac{i^2}{mn}\int\frac{d\phi}{\sqrt{\text{I}}}-\frac{2}{\sqrt{mn}}\int\frac{d\tau}{\cos^3\tau}+\frac{\sin\phi\cos\phi\sqrt{\text{I}}}{\text{MN}}. \quad . \quad (164.)$$

From (155.) we may deduce $\sin\tau=\frac{\sqrt{mn}\sin\varphi\cos\varphi}{\sqrt{\mathrm{I}}}$; . . . . (165.)

we shall therefore have $\tan\tau\sec\tau=\frac{\sqrt{mn}\sin\varphi\cos\varphi\sqrt{\mathrm{I}}}{\mathrm{MN}}$. . . . . (166.)

It may easily be shown that $\tan\tau\sec\tau$ represents the portion of a tangent to a parabola intercepted between the point of contact and the perpendicular from the focus.

Hence $\tan\tau\sec\tau=2\int\frac{d\tau}{\cos^3\tau}-\int\frac{d\tau}{\cos\tau}$. . . . . (167.)

Combining (164.), (166.) and (167.), and using the ordinary notation of elliptic integrals,

$$\left(\frac{1-n}{n}\right)\Pi_c(n,\varphi)+\left(\frac{1-m}{m}\right)\Pi_c(m,\varphi)=\frac{c^2}{mn}\mathrm{F}_c(\varphi)-\frac{1}{\sqrt{mn}}\int\frac{d\tau}{\cos\tau},$$

or as $$\frac{d\tau}{\cos\tau}=\frac{d\sin\tau}{1-\sin^2\tau};\quad \frac{1}{\sqrt{mn}}\int\frac{d\tau}{\cos\tau}=\frac{1}{\sqrt{mn}}\int\frac{\frac{d}{d\varphi}\left[\frac{\sqrt{mn}\sin\varphi\cos\varphi}{\sqrt{1-i^2\sin^2\varphi}}\right]d\varphi}{1-\left[\frac{\sqrt{mn}\sin\varphi\cos\varphi}{\sqrt{1-i^2\sin^2\varphi}}\right]^2}, \quad . . . . (168.)$$

we have therefore

$$\left(\frac{1-n}{n}\right)\Pi_c(n,\varphi)+\left(\frac{1-m}{m}\right)\Pi_c(m,\varphi)=\frac{c^2}{mn}\mathrm{F}_c(\varphi)-\frac{1}{\sqrt{mn}}\int\frac{\frac{d}{d\varphi}\left[\frac{\sqrt{mn}\sin\varphi\cos\varphi}{\sqrt{1-c^2\sin^2\varphi}}\right]d\varphi}{1-\left[\frac{\sqrt{mn}\sin\varphi\cos\varphi}{\sqrt{1-c^2\sin^2\varphi}}\right]^2}. \quad . . (169.)$$

This is the expression given by LEGENDRE, *Traité des Fonctions Elliptiques*, tom. i. p. 68. Written in the notation adopted in this paper, the formula would be

$$\left(\frac{1-n}{n}\right)\int\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}}+\left(\frac{1-m}{m}\right)\int\frac{d\varphi}{\mathrm{M}\sqrt{\mathrm{I}}}=\frac{i^2}{mn}\int\frac{d\varphi}{\sqrt{\mathrm{I}}}-\frac{1}{\sqrt{mn}}\int\frac{d\tau}{\cos\tau}. \quad . . . . (170.)$$

XLI. We may express a and b, the semiaxes of the elliptic base of the cylinder, in terms of m and n, the conjugate parameters of the elliptic integrals in the preceding equations. From the equation of condition $m+n-mn=i^2$, and (130.) we may eliminate i^2, and get

$$\frac{a^2}{k^2}=\frac{mn(1-m)}{(n-m)^2};\quad \frac{b^2}{k^2}=\frac{mn(1-n)}{(n-m)^2}. \quad . . . . (171.)$$

Therefore $$\frac{b}{a}=\sqrt{\frac{1-n}{1-m}}=\frac{\sqrt{(1-n)(1-m)}}{(1-m)}=\frac{\sqrt{1-i^2}}{1-m}=\frac{j}{1-m}.$$

Hence the ratio of the axes of the elliptic base of the cylinder is a function of the modulus and parameter.

The ratio of the corresponding quantities in the case of the spherical ellipse may be derived from the equation

$$\frac{a^2-b^2}{a^2}=i^2;\ \text{or}\ \frac{b}{a}=\sqrt{1-i^2}=j.$$

This ratio is therefore independent of the parameter. There is then an important difference in the two cases. In the one case, the ratio of the axes is independent of the parameter, and will continue invariable, while the parameter passes through every

stage of magnitude. But in the logarithmic ellipse the vertical cylinder will change its base with the change of the parameter. We shall see the importance of this remark presently.

These ratios are—

In the sphere $\frac{b}{a}=j$. In the paraboloid $\frac{b}{a}=\frac{j}{1-m}$. . . . . (172.)

XLII. Resuming equation (157.) and developing it by a process similar to that applied to (127.), we get

$$\Sigma=\frac{\alpha\gamma}{k\sqrt{\gamma(\alpha+\beta)}}\int\frac{[1-i^2\sin^2\phi]d\phi}{\left[1+\frac{\beta}{\alpha+\beta}\sin^2\phi\right]^2\sqrt{1-i^2\sin^2\phi}}-k\int\frac{d\tau}{\cos^3\tau}. \quad (173.)$$

Now (151.) and (152.) give

$$\frac{\beta}{\alpha+\beta}=m,\ \alpha\gamma=a^2k^2,\ \sqrt{\gamma(\alpha+\beta)}=\frac{k^2\sqrt{mn}}{(n-m)},\ \text{and}\ a^2=\frac{k^2mn(1-m)}{(n-m)^2}.$$

Making these substitutions, we get

$$\Sigma=a\sqrt{(1-m)}\int\frac{[1-i^2\sin^2\phi]d\phi}{[1-m\sin^2\phi]^2\sqrt{1-i^2\sin^2\phi}}-k\int\frac{d\tau}{\cos^3\tau}, \quad (174.)$$

Now let $m=0$, then (165.) gives $\tau=0$, and we shall have

$$\Sigma=a\int d\phi\sqrt{1-i^2\sin^2\phi}.$$

This is the common expression for the rectification of a plane ellipse, whose greater semiaxis is a, and eccentricity i. This is case IV. of the Table, p. 316.

We cannot arrive at this limiting expression by making $e^2=m=0$ in (53.); for this supposition would render $i=0$, which, throughout these investigations, is assumed to be invariable.

XLIII. If, as in the case of the spherical parabola, we make $n=m$, or $n=1-\sqrt{1-i^2}$, the values of $\frac{a}{k}$ and $\frac{b}{k}$ become infinite. What, then, is the meaning of the elliptic integral of the logarithmic form of the third order, when $n=m$, or $n=1-\sqrt{1-i^2}$? In the circular form of the third order, when $m=n$, $n=i$, and the spherical ellipse becomes the spherical parabola, which, as we know, may be rectified by an elliptic integral of the first order. Not only do the ratios $\frac{a}{k}$, $\frac{b}{k}$ become infinite, but they become equal, for $\frac{b^2}{a^2}=\frac{1-n}{1-m}=1$, when $m=n$. What, then, does the integral in this case signify? It does not become imaginary or change its species.

Resuming the equation established in (133.),

$$\frac{2[2n-i^2-n^2]}{\sqrt{n(1-n)(i^2-n)}}\frac{\Sigma}{k}=-n\Phi_n+\frac{[2n-i^2-n^2]}{n}\int\frac{d\phi}{N\sqrt{I}}+\left(\frac{i^2-n}{n}\right)\int\frac{d\phi}{\sqrt{I}}+\int d\phi\sqrt{I}.$$

If we now introduce the relation given in (130.) $\frac{a}{k}=\frac{\sqrt{n(i^2-n)(1-i^2)}}{2n-i^2-n^2}$, we shall have by substitution

$$\frac{2\sqrt{1-i^2}}{\sqrt{1-n}}\frac{\Sigma}{a}=-n\Phi_n+\left(\frac{2n-i^2-n^2}{n}\right)\int\frac{d\varphi}{N\sqrt{I}}+\left(\frac{i^2-n}{n}\right)\int\frac{d\varphi}{\sqrt{I}}+\int d\varphi\sqrt{I}. \quad . \quad . \quad . \quad . \quad (175.)$$

If we now suppose $m=n$, or $n=1-\sqrt{1-i^2}$, or $2n-i^2-n^2=0$, the last equation will become

$$2\sqrt{j}\,\frac{\Sigma}{a}=j\int\frac{d\varphi}{\sqrt{I}}+\int d\varphi\sqrt{I}-n\Phi_n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (176.)$$

In this case
$$\Phi=\frac{\tan\varphi\sqrt{I}}{1+j\tan^2\varphi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (177.)$$

This is the expression for the length of an arc of a logarithmic ellipse, the intersection of a cylinder, now become circular, with a paraboloid whose semiparameter $k=0$; therefore the dimensions of the paraboloid being indefinitely diminished in magnitude, this intersection of a finite circular cylinder by a paraboloid indefinitely attenuated, must take place at an infinite altitude. We naturally should suppose that the section of a cylinder which indefinitely approaches in its limit to a circular cylinder, by a paraboloid of revolution, would be a circle; yet the fact is not so. The intersection of these surfaces, instead of being a circle, is a logarithmic ellipse, whose rectification may be effected by an elliptic integral of the second order, as we shall now proceed to show.

In the first place let us conceive the paraboloid as of definite magnitude, and the cylinder to be elliptical; its semiaxes as before being a and b. Then as a and b are the ordinates of a parabola, at the points where the elliptic cylinder meets the paraboloid, at its greatest and least distances from the axis of the surfaces, we shall have

$$a^2=2kz', \quad b^2=2kz''. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (178.)$$

Hence $a^2-b^2=2k(z'-z'')$. Let $z'-z''=h$, then h is the thickness or height of that portion of the cylinder within which the logarithmic ellipse is contained.

Now (171.) gives $\quad a^2-b^2=\frac{k^2mn}{n-m}; \quad \therefore 2h=\frac{kmn}{n-m}$,

and we have also $\quad a=\frac{k\sqrt{mn(1-m)}}{n-m}, \quad$ hence $h=\frac{a}{2}\,\frac{\sqrt{mn}}{\sqrt{1-m}}$.

Now when $n=m$, $a=b$, $k=0$, while we get for h

$$h=\frac{a}{2}\,\frac{n}{\sqrt{1-n}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (179.)$$

We thus arrive at this most remarkable result, that though the cylinder changes from elliptic to circular, while the parameter of the paraboloid approximates to its limiting value 0, yet the thickness of the zone, that is h, does not also indefinitely diminish, but assumes the limiting value given above.

Now if we cut this circular cylinder, the radius of whose base is a, by a plane

making with the plane of the circular section, or with the plane of xy, an angle whose tangent is $\frac{h}{a}$, the semiaxes A and B of this plane section will manifestly be

$$B=a, \text{ and } A=\sqrt{a^2+h^2} \text{ or } A=\frac{a(2-n)}{2\sqrt{1-n}}. \quad \ldots \ldots \quad (180.)$$

If we denote the eccentricity of this plane ellipse by

$$i_{\prime},\ i_{\prime}=\frac{n}{2-n}=\frac{1-\sqrt{1-i^2}}{1+\sqrt{1-i^2}}, \text{ or writing } j \text{ for } \sqrt{1-i^2},\ i_{\prime}=\frac{1-j}{1+j}. \quad \text{Hence } j=\frac{1-i_{\prime}}{1+i_{\prime}}. \quad (181.)$$

It is shown in every treatise on elliptic integrals, (see HYMER's Integral Calculus, p. 220,) that if c and $c_{\prime}$ are two moduli connected by the equation

$$c_{\prime}=\frac{1-\sqrt{1-c^2}}{1+\sqrt{1-c^2}}=\frac{1-b}{1+b}, \quad \ldots \ldots \quad (182.)$$

and φ and ψ two angles related, as in (63.), so that

$$\tan(\psi-\varphi)=b\tan\varphi, \quad \ldots \ldots \quad (183.)$$

we shall have
$$(1+c_{\prime})E_c(\varphi)=E_{c_{\prime}}(\psi)+c_{\prime}\sin\psi-\frac{1}{2}b_{\prime}^2F_{c_{\prime}}(\psi).$$

Now
$$1+c_{\prime}=\frac{2}{1+b}, \quad b_{\prime}^2=1-c_{\prime}^2=\frac{4b}{(1+b)^2}, \text{ hence}$$

$$E_c(\varphi)=\frac{(1+b)}{2}E_{c_{\prime}}(\psi)+\frac{(1-b)}{2}\sin\psi-\frac{b}{1+b}F_{c_{\prime}}(\psi), \quad \ldots \ldots \quad (184.)$$

and, using the common notation for the present, (74.) gives

$$bF_c(\varphi)=\frac{b}{1+b}F_{c_{\prime}}(\psi). \quad \text{Adding these equations, we get}$$

$$E_c(\varphi)+bF_c(\varphi)=\frac{(1+b)}{2}E_{c_{\prime}}(\psi)+\frac{(1-b)}{2}\sin\psi, \quad \ldots \ldots \quad (185.)$$

or, using the notation adopted in this paper,

$$\int d\varphi\sqrt{\mathrm{I}}+j\int\frac{d\varphi}{\sqrt{\mathrm{I}}}=\frac{(1+j)}{2}\int d\psi\sqrt{\mathrm{I}_{\prime}}+\frac{n}{2}\sin\psi, \quad \ldots \ldots \quad (186.)$$

since
$$n=1-b=1-j.$$

Substituting the value of the first member of this equation in (176.), the resulting equation will become

$$2\sqrt{j}\frac{\Sigma}{a}=\frac{(1+j)}{2}\int d\psi\sqrt{\mathrm{I}_{\prime}}+\frac{n}{2}\sin\psi-n\frac{\sin\varphi\cos\varphi\sqrt{\mathrm{I}}}{\cos^2\varphi+j\sin^2\varphi}. \quad \ldots \ldots \quad (187.)$$

Having put for Φ_n its value in this case, namely,

$$\Phi_n=\frac{\sin\varphi\cos\varphi\sqrt{\mathrm{I}}}{\cos^2\varphi+j\sin^2\varphi},$$

we must now combine the last two members of this equation. Adding, they become

$$\frac{n}{2}\left\{\sin\psi-\frac{2\sin\varphi\cos\varphi\sqrt{\mathrm{I}}}{\cos^2\varphi+j\sin^2\varphi}\right\} \quad \ldots \ldots \quad (188.)$$

From this expression we must eliminate the functions of φ.

Now (73.) gives $$\sqrt{\mathrm{I}}=\frac{\cos^2\varphi+j\sin^2\varphi}{\sqrt{\mathrm{I}'}}, \qquad (189.)$$
writing φ for μ.

Substituting this value of $\sqrt{\mathrm{I}}$ in the preceding expression, for which we put t, we get
$$t=\frac{n}{2}\left\{\sin\psi-\frac{2\sin\varphi\cos\varphi}{\sqrt{\mathrm{I}_{,}}}\right\}. \qquad (190.)$$

From this equation we must eliminate $\sin\varphi$, $\cos\varphi$.

If we solve the preceding equation (189.) we shall obtain the resulting expressions
$$\left.\begin{aligned}2\sin^2\varphi&=1-\sqrt{\mathrm{I}_{,}}\cos\psi+i_{,}\sin^2\psi\\ 2\cos^2\varphi&=1+\sqrt{\mathrm{I}_{,}}\cos\psi-i_{,}\sin^2\psi\end{aligned}\right\} \qquad (191.)$$

Multiplying these equations together, and recollecting that $\mathrm{I}_{,}=1-i_{,}^2\sin^2\psi$, we find
$$4\cos^2\varphi\sin^2\varphi=\sin^2\psi[\mathrm{I}_{,}+2\sqrt{\mathrm{I}_{,}}i_{,}\cos\psi+i_{,}^2\cos^2\psi]. \qquad (192.)$$

Now the second member of this equation is a perfect square,

whence $$2\sin\varphi\cos\varphi=\sin\psi[\sqrt{\mathrm{I}_{,}}+i_{,}\cos\psi]. \qquad (193.)$$

Substituting this value of $2\sin\varphi\cos\varphi$ in (190.), we get
$$t=\frac{n}{2}\sin\psi\left[1-\frac{\sqrt{\mathrm{I}_{,}}+i_{,}\cos\psi}{\sqrt{\mathrm{I}_{,}}}\right]=-\frac{n}{2}\frac{i_{,}\sin\psi\cos\psi}{\sqrt{\mathrm{I}_{,}}}. \qquad (194.)$$

As $n=1-j$, and $i_{,}=\frac{1-j}{1+j}$, $n=\frac{2i_{,}}{1+i_{,}}$, equation (187.) may now be written
$$2\Sigma=\frac{a}{2}\left(\frac{1+j}{\sqrt{j}}\right)\int d\psi\sqrt{\mathrm{I}_{,}}-\frac{ai_{,}^2\sin\psi\cos\psi}{\sqrt{j}(1+i_{,})\sqrt{\mathrm{I}_{,}}}. \qquad (195.)$$

Now as $$\mathrm{A}=\frac{a}{2}\frac{(2-n)}{\sqrt{1-n}}=\frac{a}{2}\frac{(1+j)}{\sqrt{j}} \text{ and } 1+i_{,}=\frac{2}{1+j},$$

we get ultimately $$2\Sigma=\mathrm{A}\int d\psi\sqrt{\mathrm{I}_{,}}-\mathrm{A}\frac{i_{,}^2\sin\psi\cos\psi}{\sqrt{1-i_{,}^2\sin^2\psi}}. \qquad (196.)$$

The second term of the last member of this equation is evidently the common expression for a portion of a tangent to a plane ellipse between the point of contact and the foot of a perpendicular on it from the centre; while $\mathrm{A}\int d\psi\sqrt{\mathrm{I}_{,}}$, or $\mathrm{A}\int d\psi\sqrt{1-i_{,}^2\sin^2\psi}$, is the expression for the arc of a plane ellipse whose semitransverse axis is A, and eccentricity $i_{,}$.

When the function is complete, $\varphi=\frac{\pi}{2}$ and $\psi=\pi$. See (183.)

Hence as $\int_0^{\pi} d\psi\sqrt{\mathrm{I}_{,}}=2\int_0^{\frac{\pi}{2}} d\psi\sqrt{\mathrm{I}_{,}}$,
$$\Sigma=\mathrm{A}\int_0^{\frac{\pi}{2}} d\psi\sqrt{\mathrm{I}_{,}}. \qquad (197.)$$

Σ therefore, in this case, is equal to a quadrant of the plane ellipse whose principal semiaxis A, and eccentricity $i_{\prime}$ are given by the equations

$$A=\sqrt{a^2+h^2}, \text{ and } i_{\prime}=\frac{1-\sqrt{1-i^2}}{1+\sqrt{1-i^2}}. \quad (198.*)$$

To distinguish this variety of the curve, we may call it the *circular logarithmic ellipse*, as it is a section of a circular cylinder. Accordingly, in the two forms of the third order, when the conjugate parameters are equal, or $m=n$, the representative curves of those forms become the spherical parabola, and the circular logarithmic ellipse.

This is Case V. in the Table, p. 316. The results of the preceding investigation will reappear in the demonstration of the theorem, that quadrants of the spherical or logarithmic ellipse may be expressed by the help of integrals of the first and second orders.

XLIV. It is not difficult to show that this particular case of the logarithmic form, when the parameters m and n are equal, represents the curve of intersection of a circular cylinder, by a paraboloid whose principal sections are unequal.

Let
$$x^2+y^2=a^2, \text{ and } \frac{x^2}{k}+\frac{y^2}{k'}=2z \quad (199.)$$

be the equations of the circular cylinder and of the paraboloid.

Assume
$$x=a\cos\theta, \quad y=a\sin\theta. \quad (200.)$$

Then
$$2z=a^2\left\{\frac{\cos^2\theta}{k}+\frac{\sin^2\theta}{k'}\right\},$$

and
$$\frac{dx}{d\theta}=-a\sin\theta, \ \frac{dy}{d\theta}=a\cos\theta; \ \frac{dz}{d\theta}=a^2\left(\frac{1}{k'}-\frac{1}{k}\right)\sin\theta\cos\theta. \quad (201.)$$

Hence
$$\frac{d\Sigma}{d\theta}=a\left[1+a^2\left(\frac{1}{k'}-\frac{1}{k}\right)^2\sin^2\theta\cos^2\theta\right]^{\frac{1}{2}}. \quad (202.)$$

Now we may reduce this expression by two different methods to the form of an elliptic integral.

By the first method, eliminating $\cos^2\theta$, this expression becomes

$$\frac{d\Sigma^2}{d\theta^2}=a^2+a^4\left(\frac{1}{k'}-\frac{1}{k}\right)^2\sin^2\theta-a^4\left(\frac{1}{k'}-\frac{1}{k}\right)^2\sin^4\theta. \quad (203.)$$

We may, as in (124.), reduce this expression to the form of a product of two quadratic factors,

$$(A+B\sin^2\theta)(C-B\sin^2\theta)=AC+B(C-A)\sin^2\theta-B^2\sin^4\theta. \quad (204.)$$

Comparing this expression with the preceding,

$$AC=a^2, \ B=a^2\left(\frac{1}{k'}-\frac{1}{k}\right), \ C-A=a^2\left(\frac{1}{k'}-\frac{1}{k}\right) \text{ or } C=A+B, \text{ and } AC=A^2+AB=a^2. \quad (205.)$$

* Professor STOKES of Cambridge has pointed out to me, that this curve, like the plane ellipse, when the cylinder is developed on a plane, becomes a curve of sines.

Let us now, as in (126.), assume $\tan^2\theta=\frac{A}{A+B}\tan^2\varphi$; . . . . (206.)

and following the steps there indicated, we shall have

$$\Sigma=A\int\frac{d\varphi\left[1-\frac{B(2A+B)}{(A+B)^2}\sin^2\varphi\right]}{\left[1-\frac{B}{A+B}\sin^2\varphi\right]^2\sqrt{1-\frac{B(2A+B)}{(A+B)^2}\sin^2\varphi}}, \quad (207.)$$

an expression of the same form as (127.).

Let $$\frac{B}{A+B}=n,\ \frac{B(2A+B)}{(A+B)^2}=i^2; \quad (208.)$$

therefore $$1-n=\frac{A}{A+B},\ \text{and}\ 1-i^2=\frac{A^2}{(A+B)^2}$$

Hence $$1-n=\sqrt{1-i^2},\ \text{or}\ n=m \quad (209.)$$

If we develope this integral by the method indicated in XXXVI., the coefficient $\frac{2n-i^2-n^2}{n}$ of the integral $\int\frac{d\varphi}{(1-n\sin^2\varphi)\sqrt{1-i^2\sin^2\varphi}}$, in the result will be 0, and the reduced integral will become, as

$$\frac{B}{A+B}=n,\ B=\frac{nA}{1-n},\ \text{and}\ B=a^2\left(\frac{1}{k'}-\frac{1}{k}\right). \quad (210.)$$

$$\Sigma=\frac{nA}{2(1-n)}\left[\frac{1}{n}\int d\varphi\sqrt{I}+\left(\frac{1-n}{n}\right)\int\frac{d\varphi}{\sqrt{I}}-\Phi\right]. \quad (211.)$$

Let z' and z'' be the altitudes of the points above the plane of xy, in which the principal sections of the paraboloid meet the circular cylinder. Then $z''-z'$ is the height or thickness of the zone of the cylinder on which the curve is traced.

Now $$a^2=2kz',\ a^2=2k'z'',\ \text{whence}\ z''-z'=\frac{a^2}{2}\left(\frac{1}{k'}-\frac{1}{k}\right).$$

Let this altitude or thickness of the zone be put h, and we shall have

$$\Sigma=h\left[\frac{1}{n}\int d\varphi\sqrt{I}+\left(\frac{1-n}{n}\right)\int\frac{d\varphi}{\sqrt{I}}-\Phi\right]. \quad (212.)$$

Hence the arc of this species of logarithmic ellipse may be expressed by integrals of the first and second orders.

It is not a little remarkable that whether the integrals of the third order be circular or logarithmic, or, looking to their geometrical origin, spherical or parabolic, when the conjugate parameters are equal, or $m=n$, we may express the arcs of the hyperconic sections thus represented, in terms of integrals of the first and second orders only; the integral of the third order being in this case eliminated.

If we now resume equation (202.) and make

$$2\theta=\frac{\pi}{2}+\psi, \quad (213.)$$

$\sin 2\theta = 2\sin\theta\cos\theta = \cos\psi$, and $2d\theta = d\psi$. Therefore (202.) will become

$$\frac{4d\Sigma^2}{d\psi^2} = a^2 + \frac{a^4}{4}\left(\frac{1}{k'} - \frac{1}{k}\right)^2 \cos^2\psi, \quad \ldots \ldots \ldots \quad (214.)$$

hence as $h = \frac{a^2}{2}\left(\frac{1}{k'} - \frac{1}{k}\right)$, we shall have $2\Sigma = \sqrt{a^2+h^2}\int d\psi \sqrt{1 - \frac{h}{\sqrt{a^2+h^2}}\sin^2\psi}$. . (215.)

This is the common form for the rectification of a plane ellipse, whose principal semi-axes are $\sqrt{a^2+h^2}$ and a. Let $i_{,}$ be the eccentricity of this plane ellipse,

$$i_{,} = \frac{h}{\sqrt{a^2+h^2}} = \frac{B}{2A+B} = \frac{n}{2-n} = \frac{1-\sqrt{1-i^2}}{1+\sqrt{1-i^2}}, \quad \ldots \ldots \quad (216.)$$

and the relation between ϕ and ψ is given by the equations

$$2\theta = \frac{\pi}{2} + \psi, \quad \tan^2\theta = \frac{A}{A+B}\tan^2\phi, \text{ or } \tan\theta = \sqrt{1-n}\tan\phi.$$

Hence
$$\frac{1+\sin\psi}{1-\sin\psi} = (1-n)\tan^2\phi. \quad \ldots \ldots \ldots \quad (217.)$$

When $\psi = 0$, $\tan\phi = \frac{1}{\sqrt{j}}$; when $\psi = \frac{\pi}{2}$, $\phi = \frac{\pi}{2}$; when $\psi = -\frac{\pi}{2}$, $\phi = 0$. Hence ψ is measured from the perpendicular on the tangent to the ellipse, at the point which divides the elliptic quadrant into two segments whose difference is equal to $a-b$, as will be shown further on: while ϕ is measured from the semitransverse axe a. Thus while ψ varies from $-\frac{\pi}{2}$ (that is from the position at right angles to this perpendicular, and *below* it,) to 0, that is to the perpendicular itself, ϕ varies from 0 to $\tan^{-1}\frac{1}{\sqrt{j}}$; and while ψ varies from 0 to $\frac{\pi}{2}$, ϕ varies from $\tan^{-1}\frac{1}{\sqrt{j}}$ to $\frac{\pi}{2}$. Thus while ψ passes over two right angles, ϕ passes over one right angle.

We may now equate the two expressions (211.) and (215.),

$$\int d\psi\sqrt{1-i_{,}^2\sin^2\psi} = \frac{2h}{\sqrt{a^2+h^2}}\left[\frac{1}{n}\int d\phi\sqrt{\mathrm{I}} + \frac{(1-n)}{n}\int\frac{d\phi}{\sqrt{\mathrm{I}}} - \Phi\right], \quad \ldots \quad (218.)$$

or we may express an elliptic integral of the first order by means of two elliptic integrals of the second order. Thus we obtain the geometrical origin of this well-known theorem.

When the functions are complete, since

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi\sqrt{1-i_{,}^2\sin^2\psi} = 2\int_0^{\frac{\pi}{2}} d\psi\sqrt{1-i_{,}^2\sin^2\psi}, \quad \text{we get}$$

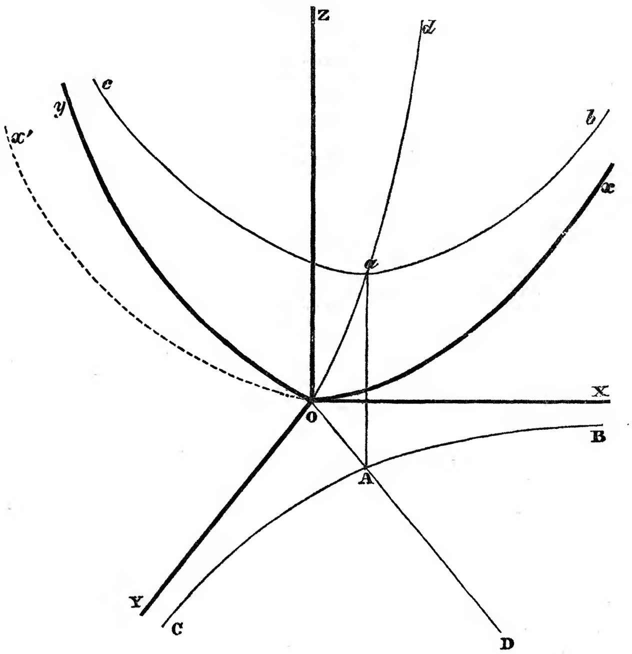
$$\int_0^{\frac{\pi}{2}} d\psi\sqrt{1-i_{,}^2\sin^2\psi} = i_{,}\left[\frac{1}{n}\int_0^{\frac{\pi}{2}} d\phi\sqrt{\mathrm{I}} + \left(\frac{1-n}{n}\right)\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{\mathrm{I}}}\right], \quad \ldots \ldots \quad (219.)$$

which agrees with (186.).

SECTION V.—*On the Logarithmic Hyperbola.*

XLV. The Logarithmic hyperbola may be defined as the curve of symmetrical intersection of a paraboloid of revolution with a right cylinder standing on a plane hyperbola as a base.

Fig. 16.



Let $Oxx_{\prime}$ be a paraboloid of revolution, whose vertex is at O, and whose axis is OZ. Let ACB be an hyperbola in the plane of xy, whose vertex is at A, whose asymptots are the lines OX, OY, and whose axis is the right line OAD. Let the planes ZOX, ZOD, ZOY cut the paraboloid in the plane parabolas Ox, Od, Oy, and let *cab* be the curve on the surface of the paraboloid whose orthogonal projection on the plane of xy is the plane hyperbola ABC. Then *acb* is the logarithmic hyperbola.

As OX is an asymptot to the hyperbolic arc AB, it is manifest that the parabolic arc Ox is a curvilinear asymptot to the arc *ab* of the logarithmic hyperbola.

Let
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \text{ and } x^2+y^2=2kz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (220.)$$
be the equations of the hyperbolic cylinder and of the paraboloid of revolution, and consequently of the curve in which they intersect. Let Υ be an arc of this curve,

then
$$\Upsilon=\int d\lambda\left[\left(\frac{dx}{d\lambda}\right)^2+\left(\frac{dy}{d\lambda}\right)^2+\left(\frac{dz}{d\lambda}\right)^2\right]^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (221.)$$
x, y, z being functions of a fourth independent variable λ.

Assume
$$x^2=\frac{a^4\cos^2\lambda}{a^2\cos^2\lambda-b^2\sin^2\lambda}, \quad y^2=\frac{b^4\sin^2\lambda}{a^2\cos^2\lambda-b^2\sin^2\lambda}*. \quad . \quad . \quad . \quad (222.)$$
It is manifest that these assumptions are compatible with the first of equation (220.), and the second of that group gives
$$x^2+y^2=\frac{a^4\cos^2\lambda+b^4\sin^2\lambda}{a^2\cos^2\lambda-b^2\sin^2\lambda}=2kz.$$

---

* We might, by the help of the imaginary transformation $\sin\theta=\sqrt{-1}\tan\theta'$, pass at once from the elliptic cylinder to the hyperbolic cylinder. Let $\tan\theta'=u$, and the resulting equation will be of the form
$$\frac{d\Upsilon}{du}=\frac{\alpha+\beta u^2+\gamma u^4}{\sqrt{A+Bu^2+Cu^4+Du^6}},$$
an expression which, on trial, it would be found very difficult to reduce. The difficulty is eluded by making the transformation pointed out and adopted in the text.

Differentiating (222.), we get

$$\left(\frac{dx}{d\lambda}\right)^2=\frac{a^4b^4\sin^2\lambda}{(a^2\cos^2\lambda-b^2\sin^2\lambda)^3},\quad \left(\frac{dy}{d\lambda}\right)^2=\frac{a^4b^4\cos^2\lambda}{(a^2\cos^2\lambda-b^2\sin^2\lambda)^3},\quad \left(\frac{dz}{d\lambda}\right)^2=\frac{(a^2+b^2)^2a^4b^4\sin^2\lambda\cos^2\lambda}{k^2(a^2\cos^2\lambda-b^2\sin^2\lambda)^4}. \quad (223.)$$

Hence $$\frac{k}{a^2b^2}\frac{d\Upsilon}{d\lambda}=\frac{[a^2k^2+(a^2+b^2)(a^2+b^2-k^2)\sin^2\lambda-(a^2+b^2)^2\sin^4\lambda]^{\frac{1}{2}}}{(a^2\cos^2\lambda-b^2\sin^2\lambda)^2}. \quad (224.)$$

Let this radical be put $=\sqrt{V}$.

Assume $V=(A+B\sin^2\lambda)(C-B\sin^2\lambda)=AC+B(C-A)\sin^2\lambda-B^2\sin^4\lambda$, (225.)

hence $AC=a^2k^2,\quad B=a^2+b^2,\quad C-A=a^2+b^2-k^2.$ (226.)

Let us now assume $\sin\varphi$ such, that

$$\sin^2\lambda=\frac{AC\sin^2\varphi}{AB+BC\cos^2\varphi}, \quad (227.)$$

then $$A+B\sin^2\lambda=\frac{A(A+C)}{A+C\cos^2\varphi},\quad C-B\sin^2\lambda=\frac{C(A+C)\cos^2\varphi}{A+C\cos^2\varphi},$$

and $$a^2\cos^2\lambda-b^2\sin^2\lambda=a^2-\frac{(a^2+b^2)AC\sin^2\varphi}{B(A+C\cos^2\varphi)};$$

or as $$a^2+b^2=B,\quad AC=a^2k^2,\quad C+k^2=A+B,$$

we get $$a^2\cos^2\lambda-b^2\sin^2\lambda=\frac{a^2(A+C)}{A+C\cos^2\varphi}\left[1-\frac{A+B}{A+C}\sin^2\varphi\right].$$

Hence $$\frac{k}{a^2b^2}\frac{d\Upsilon}{d\lambda}=\frac{\sqrt{AC}.[A+C\cos^2\varphi]\cos\varphi}{a^4(A+C)[1-l\sin^2\varphi]^2}. \quad (228.)$$

Making $$l=\frac{A+B}{A+C}. \quad (229.)$$

Differentiating the equation $\sin^2\lambda=\dfrac{AC\sin^2\varphi}{AB+BC\cos^2\varphi}$, (230.)

we get $$\frac{d\lambda}{d\varphi}=\frac{ak\sqrt{A+C}\cos\varphi}{\sqrt{B}[A+C\cos^2\varphi]\sqrt{1-\frac{C}{B}\left(\frac{A+B}{A+C}\right)\sin^2\varphi}}, \quad (231.)$$

or as $$\frac{d\Upsilon}{d\varphi}=\frac{d\Upsilon}{d\lambda}\frac{d\lambda}{d\varphi},\quad \text{making } i^2=\frac{C(A+B)}{B(A+C)}, \quad (232.)$$

we get, finally, $$\frac{\Upsilon}{k}=\frac{b^2}{\sqrt{B(A+C)}}\int\frac{\cos^2\varphi d\varphi}{[1-l\sin^2\varphi]^2\sqrt{1-i^2\sin^2\varphi}}. \quad (233.)$$

XLVI. We may develope another formula for the rectification of an arc of the logarithmic hyperbola.

Assuming the principles established in Sect. XXXVIII., we may put

$$\Upsilon=-\int p\sec\upsilon d\lambda-\int\frac{d^2p}{d\lambda^2}\sec\upsilon d\lambda. \quad (234.)$$

In this formula p is the perpendicular from the axis of the hyperbolic cylinder let fall on a tangent plane to it, passing through the element of the curve; and υ is the

angle which a tangent to this element makes with the plane of the base. υ in this equation is analogous to τ in the last section.

In the above expression, the negative sign is used as the curve is *convex* towards the origin.

Now $p^2=a^2\cos^2\lambda-b^2\sin^2\lambda$, and $\tan\upsilon=\dfrac{\frac{dz}{d\lambda}}{\sqrt{\left(\frac{dx}{d\lambda}\right)^2+\left(\frac{dy}{d\lambda}\right)^2}}$. We must substitute for these differentials, their values given in (223.), and introduce the value of φ assumed in (227.), whence

$$\sec^2\upsilon=\frac{(A+C)^2AC\cos^2\varphi}{k^2[A+C\cos^2\varphi]^2(a^2\cos^2\lambda-b^2\sin^2\lambda)}; \quad (235.)$$

$$\therefore\ p\sec\upsilon=\frac{\sqrt{AC}(A+C)\cos\varphi}{k[A+C\cos^2\varphi]}. \quad (236.)$$

But (231.) gives

$$\frac{d\lambda}{d\varphi}=\frac{\sqrt{A+C}.ak\cos\varphi}{\sqrt{B}[A+C\cos^2\varphi]\sqrt{1-i^2\sin^2\varphi}},$$

whence

$$p\sec\upsilon d\lambda=\frac{a^2k\cos^2\varphi d\varphi}{\sqrt{B(A+C)}\left[1-\frac{C}{A+C}\sin^2\varphi\right]^2\sqrt{1-i^2\sin^2\varphi}}. \quad (237.)$$

We must now determine the value of the second integral in (234.), namely,

$$\int\frac{d^2p}{d\lambda^2}\sec\upsilon d\lambda,$$

since $p^2=a^2\cos^2\lambda-b^2\sin^2\lambda$, $\quad \dfrac{d^2p}{d\lambda^2}\sec\upsilon d\lambda=-\dfrac{(a^2+b^2)[a^2\cos^4\lambda+b^2\sin^4\lambda]\sec\upsilon d\lambda}{(a^2\cos^2\lambda-b^2\sin^2)^{\frac{3}{2}}}. \quad (238.)$

Now we may derive from (223.) $\tan\upsilon=\dfrac{(a^2+b^2)\sin\lambda\cos\lambda}{k(a^2\cos^2\lambda-b^2\sin^2\lambda)^{\frac{1}{2}}}. \quad (239.)$

Differentiating this expression, then multiplying by $\sec\upsilon$, and integrating, we obtain

$$k\int\frac{d\upsilon}{\cos^3\upsilon}=(a^2+b^2)\int\frac{[a^2\cos^4\lambda+b^2\sin^4\lambda]\sec\upsilon d\lambda}{(a^2\cos^2\lambda-b^2\sin^2\lambda)^{\frac{3}{2}}}. \quad (240.)$$

Comparing this expression with (238.), and introducing into (234.) the values found in (237.) and (240.), we obtain

$$\frac{\Upsilon}{k}=\int\frac{d\upsilon}{\cos^3\upsilon}-\frac{a^2}{\sqrt{B(A+C)}}\int\frac{\cos^2\varphi d\varphi}{[1-m\sin^2\varphi]^2\sqrt{1-i^2\sin^2\varphi}}, \quad (241.)$$

making

$$m=\frac{C}{A+C}, \quad (242.)$$

since

$$l=\frac{A+B}{A+C},\ \text{and}\ i^2=\frac{C}{B}\left(\frac{A+B}{A+C}\right),\ \text{assume}\ n=\frac{i^2}{l}=\frac{C}{B} \quad (243.)$$

and we shall have m and n connected by the equation of condition, defined in (1.),

$$m+n-mn=i^2.$$

The three parameters l, m, n, and the modulus i are connected by the equations

$$ln=i^2,\quad m+n-mn=i^2. \quad (244.)$$

l and n are *reciprocal* parameters, the reader will recollect, while m and n are *conjugate* parameters.

XLVII. It was shown in (226.), that $C-A=a^2+b^2-k^2$, $B=a^2+b^2$, $k^2=A+B-C$, and $a^2k^2=AC$, whence

$$\frac{a^2}{k^2}=\frac{AC}{(A+B-C)^2}, \quad \frac{b^2}{k^2}=\frac{(A+B)(B-C)}{(A+B-C)^2}. \qquad (245.)$$

In order that these values of a and b may be real, we must have $B>C$, and A of the same sign with C, both positive, otherwise $\sqrt{V}$ in (225.) will be imaginary. As $l=\frac{A+B}{A+C}$, $l>1$; here the parameter l is greater than 1, while m and n are each less than 1.

We may express the semiaxes of the hyperbola, the base of the hyperbolic cylinder, in terms of the modulus i and the parameter l; for by the equations immediately preceding we may eliminate A, B and C in (243.). We thus find

$$\frac{a^2}{k^2}=\frac{li^2(l-1)(1-i^2)}{[l^2+i^2-2li^2]^2}, \quad \frac{b^2}{k^2}=\frac{l(l-1)(l-i^2)^2}{[l^2+i^2-2li^2]^2}; \qquad (246.)$$

therefore
$$\frac{B}{k^2}=\frac{a^2+b^2}{k^2}=\frac{l(l-1)}{l^2+i^2-2li^2}. \qquad (247.)$$

We may express the semiaxes in terms of the conjugate parameters m and n,

$$\frac{a^2}{k^2}=\frac{n^2m(1-m)}{[n+m-2mn]^2}, \quad \frac{b^2}{k^2}=\frac{m(1-n)(n+m-mn)}{(n+m-2mn)^2}; \qquad (248.)$$

hence
$$\frac{B}{k^2}=\frac{a^2+b^2}{k^2}=\frac{m}{(m+n-2mn)} \text{ and } \sqrt{B(A+C)}=\frac{\sqrt{mn}}{(m+n-2mn)}; \qquad (249.)$$

or we may express a and b more simply in terms of l and m. Eliminating n and i^2, we get
$$\frac{a^2}{k^2}=\frac{m(1-m)}{(l-m)^2}, \quad \frac{b^2}{k^2}=\frac{l(l-1)}{(l-m)^2}. \qquad (250.)$$

Let $c_{\prime}$ be the eccentricity of the hyperbolic base of the cylinder, we shall easily discover the following equation between $c_{\prime}$, i and l, analogous to (131.),

$$(c_{\prime}^2-1)i^2j^2=(l-i^2)^2. \qquad (251.)$$

Hence when i and l are given, $c_{\prime}$ may easily be found.

XLVIII. If we equate together the values found for Υ, the arc of the logarithmic hyperbola, in (233.) and (241.), we shall have

$$b^2\int\frac{\cos^2\phi d\phi}{[1-l\sin^2\phi]^2\sqrt{1-i^2\sin^2\phi}}+a^2\int\frac{\cos^2\phi d\phi}{[1-m\sin^2\phi]^2\sqrt{1-i^2\sin^2\phi}}=\sqrt{B(A+C)}\int\frac{d\upsilon}{\cos^3\upsilon}. \qquad (252.)$$

For brevity, put

$$L=1-l\sin^2\phi,\ M=1-m\sin^2\phi,\ N=1-n\sin^2\phi,\ I=1-i^2\sin^2\phi. \qquad (253.)$$

The preceding equation may now be written

$$b^2\int\frac{\cos^2\phi d\phi}{L^2\sqrt{I}}+a^2\int\frac{\cos^2\phi d\phi}{M^2\sqrt{I}}=\sqrt{B(A+C)}\int\frac{d\upsilon}{\cos^3\upsilon}; \qquad (254.)$$

or, if we substitute for the coefficients of this equation their values given in (246.), we shall have

$$(l-i^2)^2\int\frac{\cos^2\varphi d\varphi}{L^2\sqrt{I}}+i^2(1-i^2)\int\frac{\cos^2\varphi d\varphi}{M^2\sqrt{I}}=\frac{[l^2+i^2-2li^2]\sqrt{l-i^2}}{\sqrt{l(l-1)}}\int\frac{dv}{\cos^3 v}. \quad . \quad . \quad . \quad (255.)$$

Let $\delta=l^2+i^2-2li^2$, $\delta'=m^2+i^2-2mi^2$, $\Phi_l=\frac{\sin\varphi\cos\varphi\sqrt{I}}{L}$, $\Phi_m=\frac{\sin\varphi\cos\varphi\sqrt{I}}{M}$. (256.)

Now the process given in XXXVI. will enable us to develope the integrals

$$\int\frac{\cos^2\varphi d\varphi}{L^2\sqrt{I}} \quad \text{and} \int\frac{\cos^2\varphi d\varphi}{M^2\sqrt{I}}, \text{ as follows :—}$$

$$2(l-i^2)\int\frac{\cos^2\varphi d\varphi}{L^2\sqrt{I}}=l(l-i^2)\Phi_l-(l-i^2)\int d\varphi\sqrt{I}+\frac{(l-i^2)^2}{l}\int\frac{d\varphi}{\sqrt{I}}+\frac{\delta}{l}(l-i^2)\int\frac{d\varphi}{L\sqrt{I}}, \quad . \quad . \quad (257.)$$

and

$$2i^2(1-i^2)\int\frac{\cos^2\varphi d\varphi}{M^2\sqrt{I}}=-\frac{m(1-i^2)i^2}{(i^2-m)}\Phi_m+\frac{i^2(1-i^2)}{i^2-m}\int d\varphi\sqrt{I}+\frac{i^2(1-i^2)}{m}\int\frac{d\varphi}{\sqrt{I}}-\frac{\delta' i^2(1-i^2)}{m(i^2-m)}\int\frac{d\varphi}{M\sqrt{I}}. \quad (258.)$$

The equations of condition $ln=i^2$, and $m+n-mn=i^2$, give

$$\frac{i^2(1-i^2)}{i^2-m}=l-i^2, \text{ and } \frac{(l-i^2)^2}{l}+\frac{i^2(1-i^2)}{m}=\frac{(l-i^2)\delta}{l(l-1)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (259.)$$

We have also, since $m=\frac{i^2(l-1)}{l-i^2}$, $l(l-i^2)\Phi_l-\frac{mi^2(1-i^2)}{(i^2-m)}\Phi_m=\frac{\delta\sin\varphi\cos\varphi\sqrt{I}}{LM}$. . (260.)

Making these substitutions, adding together (257.) and (258.), the coefficient of $\int d\varphi\sqrt{I}$ vanishes, and we shall have

$$2(l-i^2)^2\int\frac{\cos^2\varphi d\varphi}{L^2\sqrt{I}}+2i^2(1-i^2)\int\frac{\cos^2\varphi d\varphi}{M^2\sqrt{I}}=\frac{\delta\sin\varphi\cos\varphi\sqrt{I}}{LM}+\frac{\delta(l-i^2)}{l(l-1)}\int\frac{d\varphi}{\sqrt{I}}+\frac{\delta}{l}(l-i^2)\int\frac{d\varphi}{L\sqrt{I}}-\frac{\delta'(l-i^2)}{m}\int\frac{d\varphi}{M\sqrt{I}},$$

but (255.) gives $$(l-i^2)^2\int\frac{\cos^2\varphi d\varphi}{L^2\sqrt{I}}+i^2(1-i^2)\int\frac{\cos^2\varphi d\varphi}{M^2\sqrt{I}}=\delta\sqrt{\frac{(l-i^2)}{l(l-1)}}\int\frac{dv}{\cos^3 v}.$$

Combining this equation with the preceding,

$$\frac{\delta}{l}(l-i^2)\int\frac{d\varphi}{L\sqrt{I}}-\frac{\delta'(l-i^2)}{m}\int\frac{d\varphi}{M\sqrt{I}}+\frac{\delta(l-i^2)}{l(l-1)}\int\frac{d\varphi}{\sqrt{I}}+\frac{\delta\sin\varphi\cos\varphi\sqrt{I}}{LM}=2\delta\sqrt{\frac{l-1}{l(l-1)}}\int\frac{dv}{\cos^3 v}. \quad (261.)$$

Now $$\delta'=m^2+i^2-2mi^2=\frac{i^2(1-i^2)\delta}{(l-i^2)^2}, \text{ and as } m=\frac{i^2(l-1)}{l-i^2}, \frac{\delta'(l-i^2)}{m}=\frac{(1-i^2)\delta}{(l-1)}.$$

In the last equation, substituting this value of δ', and then dividing by δ, we get

$$\frac{\sin\varphi\cos\varphi\sqrt{I}}{LM}+\frac{(l-i^2)}{l(l-1)}\int\frac{d\varphi}{\sqrt{I}}+\frac{(l-i^2)}{l}\int\frac{d\varphi}{L\sqrt{I}}-\frac{(1-i^2)}{(l-1)}\int\frac{d\varphi}{M\sqrt{I}}=2\sqrt{\frac{l-i^2}{l(l-1)}}\int\frac{dv}{\cos^3 v}. \quad . \quad . \quad (262.)$$

Now $$2\int\frac{dv}{\cos^3 v}=\tan v\sec v+\int\frac{dv}{\cos v} \text{ and } \cos^2 v=\frac{LM}{\cos^2\varphi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (263.)$$

as may be shown by combining (226.) with (235.).

Hence $$\sin v=\sqrt{\frac{l(l-1)}{l-i^2}}\tan\varphi\sqrt{I}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (264.)$$

and therefore $\tan\upsilon\sec\upsilon=\sqrt{\frac{l(l-1)}{l-i^2}}\frac{\sin\phi\cos\phi\sqrt{\mathrm{I}}}{\mathrm{LM}}$. . . . . (265.)

Substituting this value in the preceding equation, we get

$$\left(\frac{l-i^2}{l}\right)\int\frac{d\phi}{\mathrm{L}\sqrt{\mathrm{I}}}-\frac{(1-i^2)}{(l-1)}\int\frac{d\phi}{\mathrm{M}\sqrt{\mathrm{I}}}+\frac{(l-i^2)}{l(l-1)}\int\frac{d\phi}{\sqrt{\mathrm{I}}}=\sqrt{\frac{l-i^2}{l(l-1)}}\int\frac{d\upsilon}{\cos\upsilon}. \quad (266.)$$

In (170.) we showed that, m and n being conjugate parameters connected by the equation $m+n-mn=i^2$,

$$\left(\frac{1-n}{n}\right)\int\frac{d\phi}{\mathrm{N}\sqrt{\mathrm{I}}}+\frac{(1-m)}{m}\int\frac{d\phi}{\mathrm{M}\sqrt{\mathrm{I}}}=\frac{i^2}{mn}\int\frac{d\phi}{\sqrt{\mathrm{I}}}-\frac{1}{\sqrt{mn}}\int\frac{d\tau}{\cos\tau}.$$

Now $\left(\frac{1-n}{n}\right)=\frac{l}{i^2}\left(\frac{l-i^2}{l}\right)$, $\left(\frac{1-m}{m}\right)=\frac{l}{i^2}\left(\frac{1-i^2}{l-1}\right)$, $\frac{i^2}{mn}=\frac{l}{i^2}\left(\frac{l-i^2}{l-1}\right)$ and $\frac{1}{\sqrt{mn}}=\frac{l}{i^2}\sqrt{\frac{l-i^2}{l(l-1)}}$.

Substituting these values in the preceding equation, and dividing by $\frac{l}{i^2}$, we get

$$\left(\frac{l-i^2}{l}\right)\int\frac{d\phi}{\mathrm{N}\sqrt{\mathrm{I}}}+\left(\frac{1-i^2}{l-1}\right)\int\frac{d\phi}{\mathrm{M}\sqrt{\mathrm{I}}}=\left(\frac{l-i^2}{l-1}\right)\int\frac{d\phi}{\sqrt{\mathrm{I}}}-\sqrt{\frac{l-1}{l(l-1)}}\int\frac{d\tau}{\cos\tau}. \quad (267.)$$

If we add this equation to (266.), the coefficient of the integral $\int\frac{d\phi}{\mathrm{M}\sqrt{\mathrm{I}}}$ will vanish, and the resulting equation will become

$$\int\frac{d\phi}{\mathrm{L}\sqrt{\mathrm{I}}}+\int\frac{d\phi}{\mathrm{N}\sqrt{\mathrm{I}}}=\int\frac{d\phi}{\sqrt{\mathrm{I}}}+\frac{\sqrt{l}}{\sqrt{(l-1)(l-i^2)}}\left[\int\frac{d\upsilon}{\cos\upsilon}-\int\frac{d\tau}{\cos\tau}\right]. \quad (268.)$$

We shall now proceed to show that $\int\frac{d\upsilon}{\cos\upsilon}-\int\frac{d\tau}{\cos\tau}$ may be put under the form $\int\frac{d\upsilon'}{\cos\upsilon'}$, if we make the assumption $\sin\upsilon'=\frac{\sqrt{\varkappa'}\tan\phi}{\sqrt{\mathrm{I}}}$, ... (269.)

$\varkappa'$ being equal to $(1-n)\left(\frac{i^2}{n}-1\right)=\frac{(l-i^2)(l-1)}{l}$.

Now $\cos^2\upsilon=\frac{(1-m\sin^2\phi)(1-l\sin^2\phi)}{\cos^2\phi}$, as in (263.).

Hence $$\sqrt{\frac{l-i^2}{l(l-1)}}\int\frac{d\upsilon}{\cos\upsilon}=\int\frac{d\phi}{\sqrt{\mathrm{I}}}\left[\frac{[1-i^2\sin^2\phi-i^2\sin^2\phi\cos^2\phi]}{\mathrm{LM}}\right]; \quad (270.)$$

but we derive from (165.) and (166.) the value

$$\sqrt{\frac{l-i^2}{l(l-1)}}\int\frac{d\tau}{\cos\tau}=\int\frac{d\phi}{\sqrt{\mathrm{I}}}\frac{[n\cos^2\phi-n\sin^2\phi+ni^2\sin^4\phi]}{\mathrm{MN}}, \quad (271.)$$

or subtracting,

$$\sqrt{\frac{l-i^2}{l(l-1)}}\left[\int\frac{d\upsilon}{\cos\upsilon}-\int\frac{d\tau}{\cos\tau}\right]=\int\frac{\sqrt{\mathrm{I}}}{\mathrm{M}}\left[\frac{1}{\mathrm{L}}+\frac{n\sin^2\phi}{\mathrm{N}}\right]d\phi-\int\frac{\cos^2\phi}{\mathrm{M}\sqrt{\mathrm{I}}}\left[\frac{i^2\sin^2\phi}{\mathrm{L}}+\frac{n}{\mathrm{N}}\right]d\phi. \quad (272.)$$

These two latter integrals may be combined into the single integral,

$$\int\frac{[1-i^2\sin^2\phi-n\cos^2\phi]\,[1-i^2\sin^4\phi]\,d\phi}{\mathrm{LMN}\sqrt{\mathrm{I}}}. \quad (273.)$$

Now as $m+n-mn=i^2$, the first factor of the numerator becomes $(1-n)(1-m\sin^2\varphi)$ $=(1-n)\mathrm{M}$, and therefore

$$\sqrt{\frac{l-i^2}{l(l-1)}}\left[\int\frac{dv}{\cos v}-\int\frac{d\tau}{\cos\tau}\right]=\left(\frac{l-i^2}{l}\right)\int\frac{[1-i^2\sin^4\varphi]}{\mathrm{LN}\sqrt{\mathrm{I}}}. \quad \ldots \ldots \quad (274.)$$

Substituting the second member of this equation for the last in (268.), we find

$$\int\frac{d\varphi}{\mathrm{L}\sqrt{\mathrm{I}}}+\int\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}}-\int\frac{d\varphi}{\sqrt{\mathrm{I}}}=\int\frac{[1-i^2\sin^4\varphi]}{\mathrm{LN}\sqrt{\mathrm{I}}}. \quad \ldots \ldots \quad (275.)$$

Now, since we have assumed in (269.)

$$\sin v'=\frac{\sqrt{\varkappa'}\tan\varphi}{\sqrt{\mathrm{I}}},\quad \cos^2 v'=\frac{\mathrm{LN}}{\mathrm{I}\cos^2\varphi},\quad \text{hence } \frac{dv'}{\cos v'}=\frac{\sqrt{\varkappa'}[1-i^2\sin^4\varphi]d\varphi}{\mathrm{LN}\sqrt{\mathrm{I}}}; \quad \ldots \quad (276.)$$

and consequently

$$\int\frac{d\varphi}{\mathrm{L}\sqrt{\mathrm{I}}}+\int\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}}=\int\frac{d\varphi}{\sqrt{\mathrm{I}}}+\frac{1}{\sqrt{\varkappa'}}\int\frac{dv'}{\cos v'}. \quad \ldots \ldots \quad (277.)$$

This formula is usually written

$$\int\frac{d\varphi}{[1-n\sin^2\varphi]\sqrt{1-c^2\sin^2\varphi}}+\int\frac{d\varphi}{\left[1-\frac{c^2}{n}\sin^2\varphi\right]\sqrt{1-c^2\sin^2\varphi}}=\mathrm{F}_c(\varphi)+\frac{1}{\sqrt{\alpha}}\int\frac{\frac{d}{d\varphi}\left(\frac{\sqrt{\alpha}\tan\varphi}{\Delta}\right)d\varphi}{1-\left(\frac{\sqrt{\alpha}\tan\varphi}{\Delta}\right)^2}. \quad (278.)$$

We have thus shown that from the comparison of two expressions for the same arc of the logarithmic hyperbola, we may derive the well-known equation which connects two elliptic integrals of the third order, and of the logarithmic form, whose parameters are reciprocal*.

Hence also it follows that if v, τ, and v' are three normal angles, which normals to a parabola make with the axis, and if their angles are connected by the equations

$$\left.\begin{aligned}\cos^2 v &= \frac{\mathrm{ML}}{\cos^2\varphi}, & \sin v&=\sqrt{\frac{m}{n}}\tan\varphi\sqrt{\mathrm{I}},\\ \cos^2\tau &= \frac{\mathrm{MN}}{\mathrm{I}}, & \sin\tau&=\frac{\sqrt{mn}\sin\varphi\cos\varphi}{\sqrt{\mathrm{I}}},\\ \cos^2 v' &= \frac{\mathrm{LN}}{\mathrm{I}\cos^2\varphi}, & \sin v'&=\sqrt{\frac{m}{n}}(1-n)\frac{\tan\varphi}{\sqrt{\mathrm{I}}},\end{aligned}\right\} \quad \ldots \ldots \quad (279.)$$

we shall have

$$\int\frac{dv}{\cos v}=\int\frac{dv'}{\cos v'}+\int\frac{d\tau}{\cos\tau}. \quad \ldots \ldots \ldots \quad (280.)$$

* We might by the aid of the imaginary transformation $\sin\varphi=\sqrt{-1}\tan\psi$ have passed from this theorem, connecting integrals with reciprocal parameters, to the corresponding theorem in the circular form. It seems better to give an independent proof of this theorem by the method of differentiating under the sign of integration, as we shall do further on. Although these theorems have algebraically the same form, their geometrical significations are entirely different. In the logarithmic form, the theorem results from the comparison of two expressions for the *same* arc of the logarithmic hyperbola. But in the circular form, the theorem represents the sum of the arcs of two different spherical conic sections described on the same cylinder by two concentric spheres, or on the same sphere by two cylinders having their axes coincident.

Section VI.

XLIX. *The difference between an arc of a logarithmic hyperbola, and the corresponding arc of the tangent parabola, may be expressed by the arcs of a plane, a spherical and a logarithmic ellipse.*

Resuming the equation (241.), $\int\frac{\mathrm{d}\upsilon}{\cos^3\upsilon}-\frac{\Upsilon}{k}=\frac{a^2}{\sqrt{\mathrm{B}(\mathrm{A}+\mathrm{C})}}\int\frac{\cos^2\phi\mathrm{d}\phi}{\mathrm{M}^2\sqrt{\mathrm{I}}}$,

and combining (248.) with (249.), we may easily show that

$$\frac{a^2}{\sqrt{\mathrm{B}(\mathrm{A}+\mathrm{C})}}=\frac{n(1-m)\sqrt{mn}}{m+n-2nm};\quad \ldots \quad (281.)$$

and from (258.) we deduce that

$$2n(1-m)\int\frac{\cos^2\phi\mathrm{d}\phi}{\mathrm{M}^2\sqrt{\mathrm{I}}}=\frac{n}{m}(1-m)\int\frac{\mathrm{d}\phi}{\sqrt{\mathrm{I}}}+\int\mathrm{d}\phi\sqrt{\mathrm{I}}-m\Phi_m-\left(\frac{1-m}{m}\right)(m+n-2mn)\int\frac{\mathrm{d}\phi}{\mathrm{M}\sqrt{\mathrm{I}}}.$$

Let

$$\mathrm{G}=\frac{n}{m}(1-m)\int\frac{\mathrm{d}\phi}{\sqrt{\mathrm{I}}}+\int\mathrm{d}\phi\sqrt{\mathrm{I}}-m\Phi_m.\quad \ldots \quad (282.)$$

Substituting this value of $\int\frac{\cos^2\phi\mathrm{d}\phi}{\mathrm{M}^2\sqrt{\mathrm{I}}}$ in the preceding equation we get, after some obvious reductions,

$$2\int\frac{\mathrm{d}\upsilon}{\cos^3\upsilon}-\frac{2\Upsilon}{k}=\frac{\sqrt{mn}}{m+n-2mn}\mathrm{G}-\frac{n(1-m)}{\sqrt{mn}}\int\frac{\mathrm{d}\phi}{\mathrm{M}\sqrt{\mathrm{I}}}.$$

Now $a_{\prime}$ and $b_{\prime}$ being the semiaxes of the base of an elliptic cylinder whose curve of section with the paraboloid is a logarithmic ellipse, let, as in (171.),

$$\frac{a_{\prime}^2}{k^2}=\frac{mn(1-m)}{(n-m)^2},\quad \frac{b_{\prime}^2}{k^2}=\frac{mn(1-n)}{(n-m)^2};\quad \ldots \quad (283.)$$

and if we put Σ for an arc of this logarithmic ellipse, we shall have, as in (163.),

$$\frac{2\Sigma}{k}=\frac{\sqrt{mn}}{n-m}\mathrm{G}-\frac{n(1-m)}{\sqrt{mn}}\int\frac{\mathrm{d}\phi}{\mathrm{M}\sqrt{\mathrm{I}}}-2\int\frac{\mathrm{d}\tau}{\cos^3\tau}.$$

Subtracting this equation from the preceding, and replacing G by its value in (282.), we finally obtain

$$\Upsilon=k\int\frac{\mathrm{d}\upsilon}{\cos^3\upsilon}-k\int\frac{\mathrm{d}\tau}{\cos^3\tau}-\Sigma+\frac{\sqrt{mn}(1-n)mk}{(n-m)(m+n-2nm)}\mathrm{G}.\quad \ldots \quad (284.)$$

We may express the arc Υ by the help of one parabolic arc only, if we introduce the equation established in (160.), $\mathrm{S}=\Sigma+k\int\frac{\mathrm{d}\tau}{\cos^3\tau}$, hence

$$\Upsilon=k\int\frac{\mathrm{d}\upsilon}{\cos^3\upsilon}-\mathrm{S}+\frac{\sqrt{mn}(1-n)mk}{(n-m)(n+m-2mn)}\left[\frac{n}{m}(1-m)\int\frac{\mathrm{d}\phi}{\sqrt{\mathrm{I}}}+\int\mathrm{d}\phi\sqrt{\mathrm{I}}-m\Phi\right].\quad . \quad (285.)$$

When $\sin\phi=\frac{1}{\sqrt{l}}$, $\upsilon=\frac{\pi}{2}$, and the arc of the logarithmic hyperbola becomes infinite, the arc of the parabola also becomes infinite, and an asymptot to the logarithmic hyper-

bola; the difference, however, between these infinite quantities is finite, and equal to $\frac{\sqrt{mn}(1-n)mk}{(n-m)(n+m-2mn)}$G—S, integrated between the limits $\phi=0$, and $\phi=\sin^{-1}l^{-\frac{1}{2}}$.

It is needless here to dwell on the analogy which this property bears to the finite difference between the infinite arc of the common hyperbola and its asymptot. When $n=m$, the above expression becomes illusory. We shall, however, in the next article find a remarkable value for the arc of the logarithmic hyperbola, when $m=n$.

We may express the above formula somewhat more simply.

As in (248.) $\frac{b}{k}=\frac{i\sqrt{m(1-n)}}{m+n-2mn}$, and $\frac{b_{\prime}}{k}=\frac{\sqrt{mn(1-n)}}{n-m}$ $\quad \frac{bb_{\prime}}{k^2}=\frac{i}{\sqrt{m}}\,\frac{\sqrt{mn}(1-n)m}{(n-m)(n+m-2mn)}$.

The equation given in (285.) now becomes

$$\Upsilon=k\int\frac{dv}{\cos^3 v}-\text{S}+\frac{\sqrt{m}}{i}\frac{bb_{\prime}}{k}\text{G}. \quad \ldots \quad (286.)$$

The ratio between the axes of the original hyperbolic cylinder, and of the derived elliptic cylinder, may easily be determined; for

$$\frac{b^2}{a^2}=\frac{i^2(1-m)}{n^2(1-n)}, \text{ (a.)} \qquad \text{and } \frac{b^2_{\prime}}{a^2_{\prime}}=\frac{1-m}{1-n}, \text{ (b.)}.$$

Let $c_{\prime}$ be the eccentricity of the hyperbolic base, and c that of the elliptic base, then

$$n^2(c^2_{\prime}-1)=i^2(1-c^2).$$

Comparing (a.) with (b.), $\quad \sqrt{n}\,\frac{a_{\prime}}{a}=\sqrt{l}\,\frac{b_{\prime}}{b}=1+\frac{2m(1-n)}{(n-m)}$.

This equation gives at once the ratio between the axes of the hyperbolic and elliptic cylinders.

When the paraboloid becomes a plane, or when its parameter is infinite, $m=0$, S becomes an arc of a plane ellipse, $k\int\frac{dv}{\cos^3 v}$ is changed into a rectilinear asymptot, and the expression in (286.) is now transformed into $k\int\frac{dv}{\cos^3 v}-\Upsilon=\text{S}$; or the difference between the infinite branch of an hyperbola and its asymptot may be represented by an arc of a plane ellipse.

L. *On the rectification of the logarithmic hyperbola when the conjugate parameters are equal,* or $m=n$.

We have shown in XLII. that when $m=n$, the arc of the logarithmic ellipse is equivalent to an arc of a plane ellipse; so when $m=n$, the arc of a logarithmic hyperbola may be represented by an arc of a parabola, and an arc of a plane hyperbola.

In (262.), if we make $m=n$, or $l=1+j$, $n=1-j$, we shall have, writing N for M,

$$2j\int\frac{d\phi}{\text{L}\sqrt{\text{I}}}-2j\int\frac{d\phi}{\text{N}\sqrt{\text{I}}}=-\frac{2\sin\phi\cos\phi\sqrt{\text{I}}}{\text{LN}}-2\int\frac{d\phi}{\sqrt{\text{I}}}+4\int\frac{dv}{\cos^3 v}; \quad \ldots \quad \text{(a.)}$$

and in (170.), if we make $m=n$, and M=N,

$$2(1-n)\int\frac{d\phi}{\text{N}\sqrt{\text{I}}}=(2-n)\int\frac{d\phi}{\sqrt{\text{I}}}-2\int\frac{d\tau}{\cos^3\tau}+\frac{n\sin\phi\cos\phi\sqrt{\text{I}}}{\text{N}^2}. \quad \ldots \quad \text{(b.)}$$

Adding these equations together, as $1-n=j$, we get

$$2j\int\frac{d\phi}{L\sqrt{I}}=-(1-j)\int\frac{d\phi}{\sqrt{I}}+4\int\frac{dv}{\cos^3 v}-2\int\frac{d\tau}{\cos^3\tau}+\frac{\sin\phi\cos\phi\sqrt{I}}{N}\left[\frac{n}{N}-\frac{2}{L}\right]. \quad . \quad . \quad . \quad (c.)$$

Now the arc of the logarithmic hyperbola, as in (233.), is

$$\frac{\Upsilon}{k}=\frac{b^2}{\sqrt{B(A+C)}}\int\frac{\cos^2\phi d\phi}{L^2\sqrt{I}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d.)$$

In this case, the coefficient $\frac{b^2}{\sqrt{B(A+C)}}=\frac{l}{2}$, as may be shown by putting in the general value for this expression, given in (249.), $m=n$; hence

$$\frac{2\Upsilon}{k}=l\int\frac{\cos^2\phi d\phi}{L^2\sqrt{I}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e.)$$

Now (257.) gives $2(l-i^2)\int\frac{\cos^2\phi d\phi}{L^2\sqrt{I}}=l\Phi_l-\int d\phi\sqrt{I}+\left(\frac{l-i^2}{l}\right)\int\frac{d\phi}{\sqrt{I}}+\frac{\delta}{l}\int\frac{d\phi}{L\sqrt{I}}$; . . . . (f.)

and the general value of δ being $l^2+i^2-2li^2$, as in (256.), $\delta=2l(1-n)^2$, $l=2-n$, and $l-i^2=l(1-n)$, since $ln=i^2$.

The last equation may now be written, combining (e.) with it,

$$\frac{4\Upsilon}{k}=\frac{l}{1-n}\Phi_l-\frac{1}{1-n}\int d\phi\sqrt{I}+\int\frac{d\phi}{\sqrt{I}}+2j\int\frac{d\phi}{L\sqrt{I}}. \quad . \quad . \quad . \quad . \quad . \quad (287.)$$

Adding this equation to (c.),

$$\frac{4\Upsilon}{k}=4\int\frac{dv}{\cos^3 v}-2\int\frac{d\tau}{\cos^3\tau}+j\int\frac{d\phi}{\sqrt{I}}-\frac{1}{j}\int d\phi\sqrt{I}+\frac{l}{j}\Phi_l+\frac{\sin\phi\cos\phi}{N}\left[\frac{n}{N}-\frac{2}{L}\right]. \quad . \quad . \quad . \quad (288.)$$

Now $$\frac{l\Phi_l}{j}=\frac{(1+j)\sin\phi\cos\phi\sqrt{I}}{jL}=\frac{\tan\phi\sqrt{I}}{j}+\frac{\tan\phi\sqrt{I}}{L}.$$

Combining this value of Φ_l with the preceding equation, we get

$$\frac{4\Upsilon}{k}=4\int\frac{dv}{\cos^3 v}-2\int\frac{d\tau}{\cos^3\tau}+\frac{1}{j}\left[\tan\phi\sqrt{I}-\int d\phi\sqrt{I}+j^2\int\frac{d\phi}{\sqrt{I}}\right]+\tan\phi\sqrt{I}\left[\frac{n\cos^2\phi}{N^2}-\frac{2\cos^2\phi}{LN}+\frac{1}{L}\right]; \quad (28$$

and this latter term, in this case, may be reduced to $-\frac{j\tan\phi\sqrt{I}}{N^2}$.

But, a and b being the semiaxes of the hyperbolic cylinder, (248.) gives $\frac{ab}{k^2}=\frac{mnij}{(m+n-2mn)^2}$, or in this case, as $m=n$, $\frac{2\sqrt{ab}}{\sqrt{ij}}=\frac{k}{j}$.

Now $\sqrt{\frac{ab}{ij}}$ is the distance from the centre to the focus of an hyperbola, the squares of whose semiaxes are $\frac{i}{j}ab$ and $\frac{j}{i}ab$, hence

$$\frac{k}{2j}\left[\tan\phi\sqrt{I}-\int d\phi\sqrt{I}+j^2\int\frac{d\phi}{\sqrt{I}}\right]$$

represents an arc of an hyperbola the squares of whose semiaxes are $\frac{i}{j}ab$ and $\frac{j}{i}ab$.

Introduce this value of $\frac{k}{j}$, and divide by 2,

$$2\Upsilon=2k\int\frac{dv}{\cos^3 v}-k\int\frac{d\tau}{\cos^3\tau}+\sqrt{\frac{ab}{ij}}\left[\tan\varphi\sqrt{\mathrm{I}}-\int d\varphi\sqrt{\mathrm{I}}+j^2\int\frac{d\varphi}{\sqrt{\mathrm{I}}}\right]-\frac{kj\tan\varphi\sqrt{\mathrm{I}}}{2\mathrm{N}^2}. \quad . \quad (290.)$$

Now when this equation is integrated between the limits $\varphi=0$, and $\varphi=\sin^{-1}\sqrt{\frac{1}{l}}$, or, taking the corresponding values, between $\tau=0$, and $\tau=\sin^{-1}\left(\frac{1-j}{1+j}\right)$, or between $v=0$, and $v=\frac{\pi}{2}$, Υ is infinite, and the arc of the asymptotic parabola $k\int\frac{dv}{\cos^3 v}$ is also infinite, but twice the difference Δ between those infinite quantities is finite. Let $\sin^2\varphi_{,}=\frac{1}{l}$, $\sin\tau_{,}=\frac{1-j}{1+j}$, then

$$\Delta=\frac{k(1+j)^2}{8j}+k\int_0^{\tau_{,}}\frac{d\tau}{\cos^3\tau}-\sqrt{\frac{ab}{ij}}\left[1-\int_0^{\varphi_{,}}d\varphi\sqrt{\mathrm{I}}+j^2\int_0^{\varphi_{,}}\frac{d\varphi}{\sqrt{\mathrm{I}}}\right]. \quad . \quad (291.)$$

Hence the difference between the two infinite arcs of the equilateral logarithmic hyperbola, and the corresponding infinite arcs of the asymptotic parabola, is equal to a right line + an arc of a plane parabola — an arc of a plane hyperbola.

LI. On the logarithmic hyperbola, when $l=\infty$. Case XII., p. 316.

Resume (233.), or $\frac{\Upsilon}{k}=\frac{b^2}{\sqrt{\mathrm{B(A+C)}}}\int\frac{\cos^2\varphi d\varphi}{[1-l\sin^2\varphi]^2\sqrt{1-i^2\sin^2\varphi}}$.

Now as $ln=i^2$, and as i is finite, while $l=\infty$, $n=0$.

The equation of condition $m+n-mn=i^2$, gives therefore $m=i^2$. Equations (248.) and (249.) give $a=0$, $b=k$.

And as $\sqrt{\mathrm{B(A+C)}}=\frac{\mathrm{B}\sqrt{n}}{\sqrt{m}}$, we get $\frac{b^2}{\sqrt{\mathrm{B(A+C)}}}=\frac{k^2\sqrt{m}\sqrt{l}}{k^2\sqrt{n}\sqrt{l}}=\sqrt{l}$, since $m=i^2=nl$;

hence
$$\frac{\Upsilon}{k}=\sqrt{l}\int\frac{\cos^2\varphi d\varphi}{[1-l\sin^2\varphi]^2\sqrt{1-i^2\sin^2\varphi}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a.)$$

Let $l\sin^2\varphi=\sin^2\psi$, therefore $\sqrt{l}\cos\varphi d\varphi=\cos\psi d\psi$, $[1-l\sin^2\varphi]^2=\cos^4\psi$,

$$\sqrt{1-i^2\sin^2\varphi}=\sqrt{1-\frac{i^2}{l}\sin^2\psi}=\sqrt{1-n\sin^2\psi}, \text{ and } \cos\varphi=\sqrt{1-\frac{\sin^2\psi}{l}}.$$

Making these substitutions in the preceding equation, we get

$$\frac{\Upsilon}{k}=\frac{\sqrt{l}}{\sqrt{l}}\int\frac{d\psi}{\cos^3\psi}\frac{\sqrt{1-\frac{1}{l}\sin^2\psi}}{\sqrt{1-n\sin^2\psi}}. \quad \text{When } l=\infty,\ \frac{1}{l}=0,\ n=0; \quad \text{hence } \Upsilon=k\int\frac{d\psi}{\cos^3\psi}, \quad (292.)$$

or the logarithmic hyperbola in this case becomes a common parabola.

As $a=0$, $b=k$, the hyperbolic cylinder becomes a vertical plane, at right angles to the transverse axis.

Hence, comparing this result with (XIX.), we find that when the parameters are either $+\infty$ or $-\infty$, the corresponding hyperconic section is a plane principal section of the generating surface, *i. e.* either a circle or a parabola.

LII. By giving a double rectification of the common hyperbola, we shall the more readily discover the striking analogy which exists between this curve and the logarithmic hyperbola.

Let U be an arc of a common hyperbola, whose equation is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Assume $$x^2=\frac{a^4\cos^2\lambda}{a^2\cos^2\lambda-b^2\sin^2\lambda},\quad y^2=\frac{b^4\sin^2\lambda}{a^2\cos^2\lambda-b^2\sin^2\lambda}. \qquad \text{(a.)}$$

Differentiating these expressions, and substituting, we get

$$\frac{dU}{d\lambda}=\frac{b^2}{a\left[1-\frac{a^2+b^2}{a^2}\sin^2\lambda\right]^{\frac{3}{2}}}.$$ Assume $\sin^2\phi=\frac{a^2+b^2}{a^2}\sin^2\lambda$, and let $i^2=\frac{a^2}{a^2+b^2}$. . . . . (b.)

Finding from this equation the value of $\frac{d\lambda}{d\phi}$, as $\frac{dU}{d\phi}=\frac{dU}{d\lambda}\cdot\frac{d\lambda}{d\phi}$, we shall finally obtain,

since $\frac{b^2}{\sqrt{a^2+b^2}}=\frac{a(1-i^2)}{i}$, $$\frac{U}{a}=\frac{(1-i^2)}{i}\int\frac{d\phi}{[1-\sin^2\phi]\sqrt{1-i^2\sin\phi^2}}. \qquad \text{(c.)}$$

(31.) gives $-U=\int p\,d\lambda+\int\frac{d^2p}{d\lambda^2}d\lambda$, or $U=-\int p\,d\lambda-\frac{dp}{d\lambda}$. . . . . (d.)

Now as $p^2=a^2\cos^2\lambda-b^2\sin^2\lambda$, $\frac{dp}{d\lambda}=-\frac{(a^2+b^2)\sin\lambda\cos\lambda}{(a^2\cos^2\lambda-b^2\sin^2\lambda)^{\frac{1}{2}}}$, and as $\sin^2\phi=\frac{a^2}{a^2+b^2}\sin^2\lambda$, (e.)

$\frac{d\phi}{d\lambda}=\frac{\sqrt{a^2+b^2}\sqrt{1-i^2\sin^2\phi}}{a\cos\phi}$; (f.) hence $\frac{dp}{d\lambda}=-\sqrt{a^2+b^2}\tan\phi\sqrt{1-i^2\sin^2\phi}$; . . (g.)

and as $p=a\cos\phi$, $$p\,d\lambda=\frac{a^2\cos^2\phi\,d\phi}{\sqrt{(a^2+b^2)}\sqrt{1-i^2\sin^2\phi}}=\frac{a}{i}\,\frac{\{1+i^2-i^2\sin^2\phi-1\}}{\sqrt{1-i^2\sin\phi}}; \qquad \text{(h.)}$$

whence, finally, $$\frac{i}{a}U=\tan\phi\sqrt{I}-\int d\phi\sqrt{I}+(1-i^2)\int\frac{d\phi}{\sqrt{I}}. \qquad \text{(k.)}$$

This is the expression for an arc of an hyperbola referred to in (XLIX.).

The integral $$\int\frac{d\phi}{[1-i^2\sin^2\phi]^{\frac{3}{2}}}=\frac{1}{(1-i^2)}\int d\phi\sqrt{I}-\frac{i^2}{1-i^2}\,\frac{\sin\phi\cos\phi}{\sqrt{I}}.$$

See Hymer's Integral Calculus, p. 195. Adding this integral to (k),

$$\frac{i}{a}U+(1-i^2)\int\frac{d\phi}{I\sqrt{I}}=(1-i^2)\int\frac{d\phi}{\sqrt{I}}+\tan\phi\sqrt{I}-\frac{i^2\sin\phi\cos\phi}{\sqrt{I}}: \qquad \text{(m.)}$$

but $$\tan\phi\sqrt{I}-\frac{i^2\sin\phi\cos\phi}{\sqrt{I}}=\frac{(1-i^2)\tan\phi}{\sqrt{I}}.$$

Hence dividing by $(1-i^2)$, $$\frac{iU}{a(1-i^2)}+\int\frac{d\phi}{I\sqrt{I}}=\frac{\tan\phi}{\sqrt{I}}+\int\frac{d\phi}{\sqrt{I}}; \qquad \text{(n.)}$$

but (c.) gives $$\frac{iU}{a(1-i^2)}=\int\frac{d\phi}{[1-\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}.$$

Eliminating U from these equations, we obtain

$$\int\frac{d\phi}{[1-\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}+\int\frac{d\phi}{[1-i^2\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}=\int\frac{d\phi}{\sqrt{1-i^2\sin^2\phi}}+\frac{\tan\phi}{\sqrt{1-i^2\sin^2\phi}}. \qquad \text{(293.)}$$

See HYMER's Integral Calculus, p. 245. The parameters are reciprocal in this equation, being 1 and i^2.

Now this is the extreme case of the formula for the comparison of elliptic integrals of the third order and logarithmic form. We perceive that this formula results from the comparison of two expressions for the same arc of a common hyperbola. We may also see that it is the limiting case of the general formula for the comparison of elliptic integrals of the third order having reciprocal parameters; a formula which in like manner has been deduced from the comparison of two expressions for the same arc of the logarithmic hyperbola. It is also evident that $j^2 \frac{\tan\varphi}{\sqrt{I}}$ being the difference between $\tan\varphi\sqrt{I}$ and $\frac{i^2 \sin\varphi \cos\varphi}{\sqrt{I}}$, it is the difference between tangents, one drawn to the hyperbola, the other to the plane ellipse, for $\tan\varphi\sqrt{I}$ denotes the portion of a tangent to a hyperbola between the point of contact and the perpendicular on it from the centre; and $\frac{i^2 \cos\varphi \sin\varphi}{\sqrt{I}}$ denotes a similar quantity in an ellipse; this difference is precisely analogous to the expression which occurs in (284.) $\int\frac{dv}{\cos^3 v}-\int\frac{d\tau}{\cos^3\tau}$, which denotes the difference between two parabolic arcs, one drawn a tangent to the logarithmic hyperbola, the other a tangent to the logarithmic ellipse.

SECTION VII.—*On the Values of complete Elliptic Integrals of the third order.*

LIII. We have hitherto investigated the properties and lengths of elliptic curves, on the supposition that the generating surface, whether sphere or paraboloid, was invariable, and that the lengths of the curves were made up by the summation of the elements produced by the successive values given to the amplitude φ between certain limits, 0 and $\frac{\pi}{2}$, suppose, if the arcs are to be quadrants. Thus the length of the quadrant is obtained, by adding together the successive increments which result from the successive additions, indefinitely small, which are made to the amplitude. We may, however, proceed on another principle to effect the rectification of those curves. If, to fix our ideas, we want to determine the length of a quadrant of the spherical ellipse, we may imagine the vertical cylinder, which we shall suppose invariable, to be successively intersected by a series of all possible concentric spheres. Every quadrant will differ in length from the one immediately preceding it in the series, by an infinitesimal quantity; and if we take the least of these quadrants, and add to it the successive elements, by which one quadrant differs from the next immediately preceding, we shall thus obtain the length of a quadrant of the required spherical ellipse, equal to the least quadrant which can be described on the elliptic cylinder, plus the sum of all the elements between the least quadrant and the required one. Thus, for example, the least quadrant which can be drawn on an elliptic vertical cylinder, is its section by an horizontal plane, or a quadrant of the plane ellipse,

whose semiaxes are a and b. In this case the radius of the sphere is infinite. The least sphere is that whose radius is a, and which cuts the cylinder in its circular sections. Hence the greatest spherical elliptic quadrant is the quadrant of the circle whose radius is a. All the spherical elliptic quadrants will therefore be comprised between the quadrants of an ellipse, and of a circle whose radius is a. Any quadrant therefore of a given spherical ellipse is equal to a quadrant of a plane ellipse, plus a certain increment; or to a quadrant of a circle, minus a certain decrement. The same reasoning will hold as well when we take any other limits, besides 0 and $\frac{\pi}{2}$. These considerations naturally lead to the process of differentiation under the sign of integration, because we cannot express, under a finite known form, the arc of a spherical or logarithmic ellipse, and then differentiate the expression so found, with respect to a quantity which hitherto has been taken as a constant.

We may conceive the generation of successive curves of this kind to take place in another manner. Let us imagine an invariable sphere to be cut by a succession of concentric cylinders indefinitely near to each other, and generated after a given law. These cylinders will cut the sphere in a series of spherical ellipses, any one of which will differ from the one immediately preceding, by an indefinitely small quantity. If we sum all these indefinitely small quantities, or in other words, integrate the expression so found, we shall discover the finite difference between any two curves of the series separated by a finite interval. One of the limits being a known curve, the other may thus be determined.

To apply this reasoning.

In the following investigations we shall assume the generating sphere to be invariable, and the modulus i, with the amplitude φ to be constant. The intersecting cylinder we shall suppose to vary from curve to curve on the surface of the sphere. But i is constant, and $i^2=\frac{a^2-b^2}{a^2}$, see (27.). Now a and b being the semiaxes of the base of the cylinder, it follows that the bases of all the varying cylinders are concentric and similar ellipses. Hence in the elliptic integral of the third order, which represents the spherical ellipse, the parameter e^2 or m, and the criterion of sphericity $\sqrt{\varkappa}$ will vary.

In (17.) we found for a quadrant of a spherical hyperconic section, the expression

$$\sigma=\sqrt{\varkappa}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{[1-e^2\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}.$$

Let k be the radius of the sphere.

Since $e^2=\frac{k^2i^2}{k^2-j^2a^2}$, e will vary, as being a function of a the transverse semiaxe of the variable cylinder. We have also

$$e^2\varkappa=(1-e^2)(e^2-i^2).$$

Hence $$\frac{d\varkappa}{de}=-2e\left(1-\frac{i^2}{e^4}\right); \quad \text{(294.)}$$

and if, as before, we write M for $1-m\sin^2\varphi$, or $1-e^2\sin^2\varphi$, we shall have

$$\sigma=\sqrt{\varkappa}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{M\sqrt{I}}.$$

Differentiating this expression on the hypothesis that i and φ are constant, while e is variable, we shall have

$$\frac{d\sigma}{de}=\frac{1}{2}\sqrt{\varkappa}\frac{d\varkappa}{de}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{M\sqrt{I}}+\frac{\sqrt{\varkappa}}{e}2\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{M^2\sqrt{I}}-\int_0^{\frac{\pi}{2}}\frac{d\varphi}{M\sqrt{I}}\right]. \quad \text{(295.)}$$

Multiplying this equation by $\frac{\sqrt{\varkappa}}{e}$, and recollecting that $\frac{d\varkappa}{de}=-2e\left(1-\frac{i^2}{e^4}\right)$, we shall have

$$\frac{\sqrt{\varkappa}}{e}\frac{d\sigma}{de}=-\left(1-\frac{i^2}{e^4}\right)\int_0^{\frac{\pi}{2}}\frac{d\varphi}{M\sqrt{I}}+\frac{2\varkappa}{e^2}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{M^2\sqrt{I}}-\frac{2\varkappa}{e^2}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{M\sqrt{I}}. \quad \text{(296.)}$$

But (see HYMER's Integral Calculus, p. 195)

$$\frac{2\varkappa}{e^2}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{M^2\sqrt{I}}=\left[\frac{2}{e^2}(1+i^2)-1-\frac{3i^2}{e^4}\right]\int_0^{\frac{\pi}{2}}\frac{d\varphi}{M\sqrt{I}}-\left(\frac{e^2-i^2}{e^4}\right)\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{I}}+\frac{1}{e^2}\int_0^{\frac{\pi}{2}}d\varphi\sqrt{I}. \quad \text{(297.)}$$

Introducing this value into the preceding equation, the coefficient of $\int_0^{\frac{\pi}{2}}\frac{d\varphi}{M\sqrt{I}}$ will vanish, and we shall have

$$\frac{\sqrt{\varkappa}}{e}\frac{d\sigma}{de}=-\left(\frac{e^2-i^2}{e^4}\right)\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{I}}+\frac{1}{e^2}\int_0^{\frac{\pi}{2}}d\varphi\sqrt{I}. \quad \text{(298.)}$$

Dividing by $\frac{\sqrt{\varkappa}}{e}$, and integrating on the hypothesis that φ and i are constant,

$$\sigma=\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{I}\right]\int\frac{de}{e\sqrt{\varkappa}}-\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{I}}\right]\int\frac{de(e^2-i^2)}{e^3\sqrt{\varkappa}}+\text{constant};$$

or as $e\sqrt{\varkappa}=\sqrt{(1-e^2)(e^2-i^2)}$, we shall have

$$\sigma=\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{I}\right]\int\frac{de}{\sqrt{(1-e^2)(e^2-i^2)}}-\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{I}}\right]\int\frac{de}{e^2}\sqrt{\frac{e^2-i^2}{1-e^2}}+\text{constant}. \quad \text{(299.)}$$

We must recollect that the definite integrals within the brackets are functions, not of φ, but of i^2, 0, and $\frac{\pi}{2}$. They are therefore constants.

It is not a little remarkable that the coefficients of the *definite* elliptic integrals are themselves also elliptic integrals of the first and second orders. To show this, assume

$$e^2=\cos^2\theta+i^2\sin^2\theta. \quad \text{(300.)}$$

Therefore $1-e^2=j^2\sin^2\theta$, and $e^2-i^2=j^2\cos^2\theta$; we have also $ede=-j^2\sin\theta\cos\theta d\theta$.

Hence, if $1-j^2\sin^2\theta=J$, $$\int\frac{de}{\sqrt{(e^2-i^2)(1-e^2)}}=-\int\frac{d\theta}{\sqrt{1-j^2\sin^2\theta}}=-\int\frac{d\theta}{\sqrt{J}}, \quad \text{(301.)}$$

and $$\sqrt{\varkappa}=\frac{j^2\sin\theta\cos\theta}{\sqrt{1-j^2\sin^2\theta}}. \quad \text{(302.)}$$

In the same manner we may show that

$$\int\sqrt{\frac{e^2-i^2}{1-e^2}}\,\frac{de}{e^2}=-\int\frac{d\theta}{\sqrt{1-j^2\sin^2\theta}}+i^2\int\frac{d\theta}{[1-j^2\sin^2\theta]^{\frac{3}{2}}}; \quad \ldots\ldots \quad (303.)$$

but

$$i^2\int\frac{d\theta}{[1-j^2\sin^2\theta]^{\frac{3}{2}}}=\int d\theta\sqrt{1-j^2\sin^2\theta}-j^2\frac{\sin\theta\cos\theta}{\sqrt{\mathrm{J}}}. \quad \ldots\ldots \quad (304.)$$

Hence

$$\int\sqrt{\frac{e^2-i^2}{1-e^2}}\,\frac{de}{e^2}=\int d\theta\sqrt{\mathrm{J}}-\int\frac{d\theta}{\sqrt{\mathrm{J}}}-j^2\frac{\sin\theta\cos\theta}{\sqrt{\mathrm{J}}}. \quad \ldots\ldots \quad (305.)$$

Substituting these values in (141.), we obtain

$$\sigma=\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}\right]\left[\int\frac{d\theta}{\sqrt{\mathrm{J}}}-\int d\theta\sqrt{\mathrm{J}}+j^2\frac{\sin\theta\cos\theta}{\sqrt{\mathrm{J}}}\right]-\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{\mathrm{I}}\right]\int\frac{d\theta}{\sqrt{\mathrm{J}}}+\text{constant}. \quad (306.)$$

To determine this constant. We must not suppose $i=0$, in this case, as is generally done, to determine the constant. This would be to violate the supposition on which we have all along proceeded, namely, that the variable cylinders are all similar, and therefore that i must be constant. We must determine the constant from other considerations.

Since $e^2=\frac{i^2k^2}{k^2-j^2a^2}$, when $a=0$, $e^2=i^2$. But $e^2=\cos^2\theta+i^2\sin^2\theta$, therefore $\theta=\frac{\pi}{2}$. As a, the major semiaxe of the base of the cylinder, is supposed to vanish, the curve diminishes to a point, and therefore $\sigma=0$.

When $a=k$, $e^2=1$, and $\theta=0$. We have in this case $\sigma=\frac{\pi}{2}$; for the sections of a sphere by an elliptic cylinder, whose greater axis is equal to the diameter of the sphere, are two semicircles of a great circle of the sphere. Hence, when $\theta=0$, $\sigma=\frac{\pi}{2}$, $\sin\theta=0$, $\int d\theta\sqrt{\mathrm{J}}=0$, $\int\frac{d\theta}{\sqrt{\mathrm{J}}}=0$; therefore the constant is equal to σ, when $\theta=0$. But when $\theta=0$, $\sigma=\frac{\pi}{2}$, or the constant is equal to $\frac{\pi}{2}$.

The formula now becomes

$$\sigma=\frac{\pi}{2}-\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{\mathrm{I}}\right]\int\frac{d\theta}{\sqrt{\mathrm{J}}}+\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}\right]-\left[\int\frac{d\theta}{\sqrt{\mathrm{J}}}-\int d\theta\sqrt{\mathrm{J}}+j^2\frac{\sin\theta\cos\theta}{\sqrt{\mathrm{J}}}\right]. \quad \ldots \quad (307.)$$

When $\theta=\frac{\pi}{2}$, $e=i$, and $\sigma=0$, as the variable cylinder is in this case diminished to a right line; therefore the preceding formula will become, using the ordinary notation of elliptic integrals,

$$\frac{\pi}{2}=\mathrm{E}_i\mathrm{F}_j+\mathrm{E}_j\mathrm{F}_i-\mathrm{F}_i\mathrm{F}_j. \quad \ldots\ldots\ldots \quad (308.)$$

Hence we obtain the true geometrical meaning of this curious formula of verification discovered by LEGENDRE. In its general form (307.), it represents the *difference* between the quadrants of a great circle and of a spherical ellipse. When the spherical ellipse vanishes to a point, this expression must represent, as in (308.), the quadrant of a circle.

LIV. If we now apply the preceding investigations to the curve described on

the same sphere by the reciprocal cylinder, or by the cylinder which gives a function having a reciprocal parameter, we shall find

$$\sigma'=\left[\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\sqrt{\mathrm{I}}\right]\int\frac{\mathrm{d}e'}{\sqrt{(e'^2-i^2)(1-e'^2)}}-\left[\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\sqrt{\mathrm{I}}}\right]\int\sqrt{\frac{e'^2-i^2}{1-e'^2}}\frac{\mathrm{d}e'}{e'^2}+\text{constant.} \quad . \quad . \quad (309.)$$

But by the conditions of the question, as

$$ee'=i,\ e'^2=\frac{i^2}{1-j^2\sin^2\theta},\qquad \int\frac{\mathrm{d}e'}{\sqrt{(e'^2-i^2)(1-e'^2)}}=\int\frac{\mathrm{d}\theta}{\sqrt{1-j^2\sin^2\theta}}, \quad . \quad (310.)$$

and

$$\int\frac{\mathrm{d}e'}{e'^2}\sqrt{\frac{e'^2-i^2}{1-e'^2}}=\int\frac{j^2\sin^2\theta\mathrm{d}\theta}{\sqrt{1-j^2\sin^2\theta}}=\int\frac{\mathrm{d}\theta}{\sqrt{1-j^2\sin^2\theta}}-\int\mathrm{d}\theta\sqrt{1-j^2\sin^2\theta}.$$

Substituting these values of the integrals in (309.),

$$\sigma'=\left[\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\sqrt{\mathrm{I}}\right]\int\frac{\mathrm{d}\theta}{\sqrt{\mathrm{J}}}-\left[\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\sqrt{\mathrm{I}}}\right]\left[\int\frac{\mathrm{d}\theta}{\sqrt{\mathrm{J}}}-\int\mathrm{d}\theta\sqrt{\mathrm{J}}\right]+\text{constant.} \quad . \quad . \quad (311.)$$

We shall now show that the constant $=0$.

When $\theta=0$, $e=1$, and therefore $e'=i$. Since $e'=i$, and σ is a quadrant of the vanishing spherical ellipse whose principal arcs, $\alpha=0$, $\beta=0$, we shall have $\sigma=0$. Hence also $\int\mathrm{d}\theta\sqrt{\mathrm{J}}=0$, $\int\frac{\mathrm{d}\theta}{\mathrm{J}}=0$; therefore the constant is 0. When $\theta=\frac{\pi}{2}$ $e'=1$, and (309.) becomes

$$\frac{\pi}{2}=\left[\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\sqrt{\mathrm{I}}\right]\left[\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{\sqrt{\mathrm{J}}}\right]+\left[\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\sqrt{\mathrm{I}}}\right]\left[\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\sqrt{\mathrm{J}}\right]-\left[\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\sqrt{\mathrm{I}}}\right]\left[\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{\sqrt{\mathrm{J}}}\right], \quad . \quad . \quad (312.)$$

or, in the common notation, $\frac{\pi}{2}=\mathrm{E}_i\mathrm{F}_j+\mathrm{E}_j\mathrm{F}_i-\mathrm{F}_i\mathrm{F}_j$,

a formula already established in (308.).

If we add together (307.) and (312.), we shall have, since $\sqrt{\varkappa}=\frac{j^2\sin\theta\cos\theta}{\sqrt{1-j^2\sin^2\theta}}$,

$$\sigma+\sigma'=\frac{\pi}{2}+\sqrt{\varkappa}\left[\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\sqrt{\mathrm{I}}}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (313.)$$

Now $\sigma=\left(\frac{1-m}{m}\right)\sqrt{mn}\int\frac{\mathrm{d}\varphi}{[1-m\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}$, $\sigma'=\left(\frac{1-m_{\prime}}{m_{\prime}}\right)\sqrt{m_{\prime}n_{\prime}}\int\frac{\mathrm{d}\varphi}{[1-m_{\prime}\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}$,

in which $mm'=i^2$.

Whence, as $\left(\frac{1-m}{m}\right)\sqrt{mn}=\left(\frac{1-m_{\prime}}{m_{\prime}}\right)\sqrt{m_{\prime}n_{\prime}}=\sqrt{\varkappa}$, as we have shown in (113.),

$$\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{[1-m\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}+\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\left[1-\frac{i^2}{m}\sin^2\varphi\right]\sqrt{1-i^2\sin\varphi}}=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\sqrt{1-i^2\sin^2\varphi}}+\frac{\pi}{2}. \quad (314.)$$

The reader will observe how very different are the geometrical origins of two algebraical formulæ apparently similar. In the logarithmic form of the elliptic integral, the formula for the comparison of elliptic integrals, with reciprocal parameters (one of which is greater, while the other is less than 1), resulted from putting in equation

two algebraical expressions for the *same* arc of the *one* logarithmic hyperbola. See Art. XLVIII. In the preceding case, that of the spherical ellipse, the analogous formula expresses the sum of the arcs of two inverse spherical ellipses, whose amplitudes are the same.

LV. We shall use the term *inverse spherical ellipses* to denote curves whose representative elliptic integrals have *reciprocal* parameters. The terms *reciprocal* and *supplemental* have long since been appropriated to curves otherwise related.

Let α and β, $\alpha_{,}$ and $\beta_{,}$ denote the principal semiarcs of two such curves. Since the modulus i is the same in both integrals, the orthogonal projections of these curves, on the base of the hemisphere, are similar ellipses. (15.) gives

$$e^2=i^2\sec^2\beta,\ e_{,}^2=i^2\sec^2\beta_{,}, \text{ and we assume } e^2e_{,}^2=i^2.$$

Hence
$$\sec\beta\sec\beta_{,}i=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (315.)$$

Again, as $\tan^2\alpha(1-e^2)=\tan^2\beta=\sec^2\beta-1$, and $\tan^2\alpha_{,}(1-e_{,}^2)=\tan^2\beta_{,}=\sec^2\beta_{,}-1$; multiplying these expressions together, and introducing the relation in (315.),

$$\tan^2\alpha\tan^2\alpha_{,}i^2=\frac{i^2\sec^2\beta\sec^2\beta_{,}-i^2(\sec^2\beta+\sec^2\beta_{,})+i^2}{1+i^2-i^2(\sec^2\beta+\sec^2\beta_{,})}=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (316.)$$

Hence the principal arcs of the inverse spherical ellipses are connected by the symmetrical relations

$$\tan\alpha\tan\alpha_{,}\,i=1, \text{ and } \sec\beta\sec\beta_{,}i=1. \quad . \quad . \quad . \quad . \quad . \quad (317.)$$

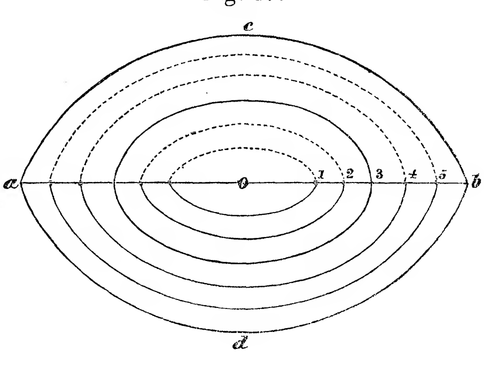
When the inverse curves coincide, $\alpha=\alpha_{,}$, $\beta=\beta_{,}$, and the last equations may be reduced to $\tan^2\alpha-\tan^2\beta=1$. Now we have shown in (59.) that when the principal arcs of a spherical hyperconic section are so related, the curve is the spherical parabola, or when the curve becomes its own inverse, it is the spherical parabola.

We have shown in (15.) that $i^2=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}=1-\frac{\sin^2\beta}{\sin^2\alpha}$, but (3.) gives $\cos\eta=\frac{\sin\beta}{\sin\alpha}$, 2η being the angle between the cyclic arcs of the spherical ellipse. Hence $i=\sin\eta$, but i is constant. Therefore all inverse spherical ellipses have the same cyclic arcs.

That portion of the surface of a sphere which lies between the cyclic circles may be called the *cyclic area.*

The spherical parabola divides the cyclic area into two regions. In the one, between the pole and the spherical parabola, lie all the inverse curves, whose parameters range from i^2 to i. In the other, between the spherical parabola and the cyclic circles, lie all the conjugate inverse curves, whose parameters range from i to 1.

Fig. 17.



Let acb, adb be the cyclic circles, the intersection of the sphere by an elliptic cylinder, whose transverse axe is equal to the diameter of the sphere, and whose minor axe is $2j$. Let a series of concyclic spherical ellipses be described within this cyclic area, whose semitransverse arcs are 01, 02, 04, 05, and let 03 be the spherical parabola of the series. For every curve, 01, or 02, *within* the spherical parabola, there may be found another *without* it,

05, or 04, such that their principal arcs are connected by the equations

$$\tan\alpha \tan\alpha_{i}=1, \quad \sec\beta \sec\beta_{i}=1.$$

The algebraic expressions for the arcs of these curves, having the same amplitude, give elliptic integrals with *reciprocal* parameters.

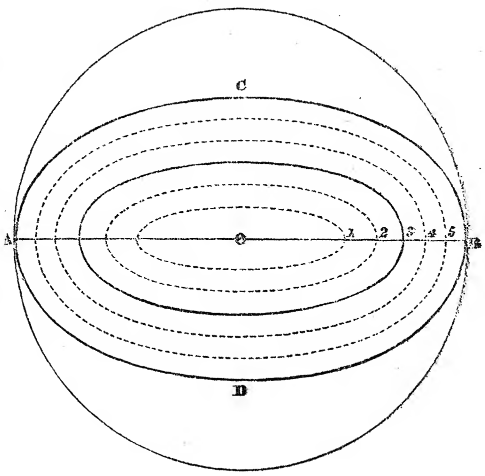


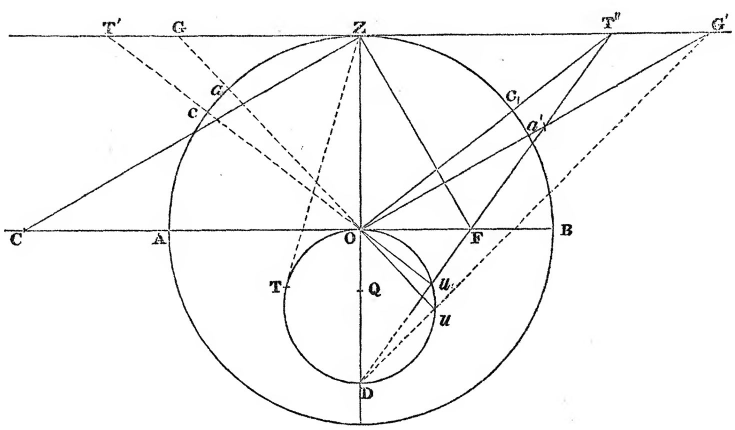
Fig. 18.

The concyclic spherical ellipses will be orthogonally projected on the base of the hemisphere into as many concentric and similar plane ellipses, whose semiaxes are 01, 02, 04, 05. The cyclic area will be projected into the plane ellipse, and the spherical parabola into the area of the plane ellipse, whose transverse semiaxe is $\frac{k}{\sqrt{1+i}}$. Let E be the area of the plane ellipse, the projection of the cyclic area, and Π the area of the plane ellipse, the projection of the spherical parabola. Then $E=\pi j$, and $\Pi=\frac{\pi j}{1+i}$, whence $\frac{E-\Pi}{\Pi}=i$, or the ellipse, the projection of the spherical parabola, divides the area of the ellipse, the projection of the cyclic area, into two portions, such that the outer is to the inner as $i:1$.

The reader must have observed the importance of this curve, the spherical parabola, in the discussion of the geometrical theory of elliptic integrals.

We may determine the principal arcs of two inverse spherical ellipses by a simple geometrical construction. Let AZB be a vertical section of the hemisphere, on which

Fig. 19.



the curves are to be described. Let F be the focus of the elliptic base of the maximum cylinder, whose principal transverse axis is accordingly equal to the diameter of the sphere. Join OZ, FZ, and draw ZC at right angles to ZF, meeting the line AO in C. Produce ZO until OD=AC, and on OD as diameter describe a circle. We are required, given one principal arc Za, to determine the corresponding principal arc Za' of the inverse hyperconic. Draw the tangent ZG. Through a draw

the line GOu. Through D draw the line $DuG_{,}$. Join $OG_{,}$, it will cut the sphere in $a_{,}$, the vertex of the principal arc $Za_{,}$. Let $OZ=k$, then $ZG=k\tan\alpha$, and as CZF is a right-angled triangle, $CO=ZD=\frac{k^2}{\sqrt{k^2-B^2}}=\frac{k}{i}$, k and B being the semiaxes of the maximum cylinder. As all the bases of the cylinders are similar, $\frac{k^2-B^2}{k^2}=\frac{a^2-b^2}{a^2}=i^2$.

Now as ZOG and ZDG$'$ are similar triangles, ZG : ZO : : ZD : ZG$'$, or $k\tan\alpha : k : : \frac{k}{i} : ZG'$, or $ZG'=\frac{k}{i\tan\alpha}$. But $ZG'=k\tan\alpha_{,}$, hence $\tan\alpha\tan\alpha_{,}i=1$, or the arcs α and $\alpha_{,}$ are connected by the equation established in (317.).

When we require to know which of these successive curves on this sphere is the spherical parabola, the same construction will enable us to determine it. Draw ZT, a tangent to the circle on OD, take $ZT'=ZT''=ZT$, and join T$'$ and T$''$ with O cutting the sphere in c and c'. $Zc=Zc'$ is the principal semitransverse arc of the spherical parabola, for $\overline{ZT}_{,}^{2}=k^2\tan^2\alpha=OZ.ZD=\frac{k^2}{i}$, or $\tan^2\alpha=\frac{1}{i}$.

As $ZT'>ZO$, $cZc'>\frac{\pi}{2}$, or the principal arc of a spherical parabola is always greater than a right angle. Since in the spherical parabola $\gamma+2\epsilon=\frac{\pi}{2}$, the angle $COT'=2\epsilon$, or COT$'$ is equal to the distance between the foci of the curve.

LVI. If we revert to the general formula (307.) and take $\breve{\sigma}$ as the quadrant of a spherical parabola, the integrations with respect to θ must take place between $\theta=0$, and $\theta_{,}=\tan^{-1}\left(\frac{1}{\sqrt{i}}\right)$, for $e^2=i$, in (300.) gives $\tan\theta=\frac{1}{\sqrt{i}}$. Hence

$$\breve{\sigma}=\frac{\pi}{2}+\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{I}}\right]\left[\int_0^{\theta_{,}}\frac{d\theta}{\sqrt{J}}\right]-\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{I}}\right]\left[\int_0^{\theta_{,}}d\theta\sqrt{J}\right]-\left[\int_0^{\frac{\pi}{2}}d\phi\sqrt{I}\right]\left[\int_0^{\theta_{,}}d\theta\sqrt{J}\right]+(1-i)\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{I}}\right]. \quad (318.)$$

Since $$\frac{j^2\sin\theta\cos\theta}{\sqrt{1-j^2\sin^2\theta}}=(1-i),\text{ when }\tan\theta=\left(\frac{1}{i}\right)^{\frac{1}{2}}.$$

Putting the sum of these integrals $=\Delta$, we shall have $\breve{\sigma}=\frac{\pi}{2}-\Delta$.

But (68.) gives for the quadrant of the spherical parabola

$$\breve{\sigma}=\frac{j^2}{(1+i)^2}\int_0^{\mu_{,}}\frac{d\mu}{\sqrt{1-\frac{4i}{(1+i)^2}\sin^2\mu}}+\frac{\pi}{4}.$$

Comparing these expressions for the same arc $\breve{\sigma}$,

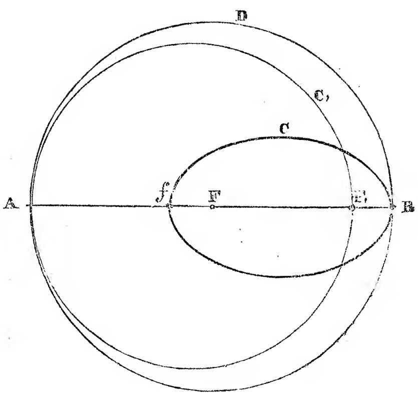
$$\frac{\pi}{4}=\frac{j^2}{(1+i)^2}\int_0^{\mu_{,}}\frac{d\mu}{\sqrt{1-\frac{4i}{(1+i)^2}\sin^2\mu}}+\Delta, \quad \ldots\ldots\ldots \quad (319.)$$

μ being taken between the limits $\mu=0$, and $\mu_{,}=\tan^{-1}\left(\frac{1}{\sqrt{j}}\right)$.

It is easy to show that the integrals of the first order in Art. LIII. may be represented

by two confocal spherical parabolas, having one common focus, and the nearer vertex of the one curve on the focus of the other. Thus let F be the pole of the hemisphere ABD. Let $\mathrm{BC}f$ and $\mathrm{AC}_{\prime}\mathrm{F}_{\prime}$ denote two spherical parabolas having one common focus at F; $\mathrm{F}_{\prime}$ and f being the other foci. Let $\mathrm{F}f=\gamma$, and therefore $\mathrm{FF}=\frac{\pi}{2}-\gamma$. Hence the modular angles of the two curves are γ, and $\frac{\pi}{2}-\gamma$, and if we make $\cos\gamma=i$, $\cos\left(\frac{\pi}{2}-\gamma\right)=j$.

Fig. 20.



Thus while the arc of the one is given by the integral $j\int\frac{d\varphi}{\sqrt{1-i^2\sin^2\varphi}}$ the arc of the other depends on the integral $i\int\frac{d\varphi}{\sqrt{1-j^2\sin\varphi^2}}$.

LVII. *On the value of the complete elliptic integral of the third order and logarithmic form.*

Let

$$\int_0^{\frac{\pi}{2}}\frac{d\varphi}{[1-n\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}=\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}}. \quad \ldots \quad (320.)$$

Assume $\varkappa$ the criterion of sphericity $=(1-n)\left(\frac{i^2}{n}-1\right)$,

then

$$\frac{d}{dn}\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}}\right]=\frac{1}{n}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\mathrm{N}^2\sqrt{\mathrm{I}}}-\frac{1}{n}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}}. \quad \ldots \quad (321.)$$

Multiply by $2\varkappa$, then

$$2\varkappa\frac{d}{dn}\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}}\right]=\frac{2\varkappa}{n}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\mathrm{N}^2\sqrt{\mathrm{I}}}-\frac{2\varkappa}{n}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}} \quad \ldots \quad (322.)$$

But (see HYMER's Integral Calculus, p. 195)

$$\frac{2\varkappa}{n}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\mathrm{N}^2\sqrt{\mathrm{I}}}=\left[1-\frac{2}{n}(1+i^2)+\frac{3i^2}{n^2}\right]\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}}-\left(\frac{i^2-n}{n^2}\right)\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}-\int_0^{\frac{\pi}{2}}d\varphi\sqrt{\mathrm{I}}, \quad \ldots \quad (323.)$$

and

$$\frac{2\varkappa}{n}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}}=\left[\frac{2i^2}{n^2}-\frac{2}{n}-\frac{2i^2}{n}+2\right]\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}}.$$

Introducing the substitutions suggested by the two latter equations into (322.),

$$2\varkappa\frac{d}{dn}\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}}\right]=\left(\frac{i^2}{n^2}-1\right)\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}}-\left(\frac{i^2-n}{n^2}\right)\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}-\frac{1}{n}\int_0^{\frac{\pi}{2}}d\varphi\sqrt{\mathrm{I}}. \quad \ldots \quad (324.)$$

Now $\frac{d\varkappa}{dn}=-\left(\frac{i^2}{n^2}-1\right)$, whence

$$2\varkappa\frac{d}{dn}\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}}\right]+\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}}\right]\frac{d\varkappa}{dn}=-\left(\frac{i^2-n}{n^2}\right)\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}-\frac{1}{n}\int_0^{\frac{\pi}{2}}d\varphi\sqrt{\mathrm{I}}. \quad \ldots \quad (325.)$$

If we divide this equation by $2\sqrt{\varkappa}$, the first member will be the differential of $\sqrt{\varkappa}\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{N\sqrt{I}}\right]$. Integrating this equation,

$$2\sqrt{\varkappa}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{N\sqrt{I}}=-\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{I}}\right]\int\frac{(i^2-n)dn}{n^2\sqrt{\varkappa}}-\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{I}\right]\int\frac{dn}{n\sqrt{\varkappa}}. \quad . \quad . \quad . \quad . \quad (326.)$$

Assume $n=i^2\sin^2\theta$, then $\varkappa=\frac{1-i^2\sin^2\theta}{\tan^2\theta}$, $dn=2i^2\sin\theta\cos\theta d\theta$. . . . . (327.)

Hence
$$\int\frac{(i^2-n)dn}{n^2\sqrt{\varkappa}}=2\int\frac{d\theta}{\tan^2\theta\sqrt{1-i^2\sin^2\theta}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (328.)$$

We must now integrate this expression,

$$\left.\begin{aligned}
\int\frac{d\theta}{\tan^2\theta\sqrt{1-i^2\sin^2\theta}} &=\int\frac{d\theta}{\sin^2\theta\sqrt{1-i^2\sin^2\theta}}-\int\frac{d\theta}{\sqrt{1-i^2\sin^2\theta}},\\
\int\frac{d\theta}{\sin^2\theta\sqrt{1-i^2\sin^2\theta}} &=-\frac{\cot\theta}{\sqrt{1-i^2\sin^2\theta}}+\int\frac{i^2\cos^2\theta d\theta}{\sqrt{(1-i^2\sin^2\theta)^{\frac{3}{2}}}}\\
\int\frac{i^2\cos^2\theta d\theta}{(1-i^2\sin^2\theta)^{\frac{3}{2}}} &=\int\frac{d\theta}{\sqrt{1-i^2\sin^2\theta}}-(1-i^2)\int\frac{d\theta}{(1-i^2\sin^2\theta)^{\frac{3}{2}}}\\
-(1-i^2)\int\frac{d\theta}{(1-i^2\sin^2\theta)^{\frac{3}{2}}} &=\frac{i^2\sin\theta\cos\theta}{\sqrt{1-i^2\sin^2\theta}}-\int d\theta\sqrt{1-i^2\sin^2\theta};
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (329.)$$

adding these equations,

$$\left.\begin{aligned}
\int\frac{d\theta}{\tan^2\theta\sqrt{1-i^2\sin^2\theta}} &=\frac{i^2\sin\theta\cos\theta}{\sqrt{1-i^2\sin^2\theta}}-\frac{\cot\theta}{\sqrt{1-i^2\sin^2\theta}}-\int d\theta\sqrt{1-i^2\sin^2\theta};\\
\therefore -\int\frac{d\theta}{\tan^2\theta\sqrt{1-i^2\sin^2\theta}} &=\cot\theta\sqrt{1-i^2\sin^2\theta}+\int d\theta\sqrt{1-i^2\sin^2\theta}\ (299.).
\end{aligned}\right\} \quad . \quad . \quad . \quad (330.)$$

We have next to compute the value of the integral $\int\frac{dn}{n\sqrt{\varkappa}}$.

Now
$$\int\frac{dn}{n\sqrt{\varkappa}}=\int\frac{d\theta}{\sqrt{1-i^2\sin^2\theta}}=\int\frac{d\theta}{\sqrt{{}_\theta I}}.$$

Substituting these values of the integrals in (326.),

$$\sqrt{\varkappa}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{N\sqrt{I}}=\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{I}}\right]\left[\cot\theta\sqrt{{}_\theta I}+\int d\theta\sqrt{{}_\theta I}\right]-\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{I}\right]\int\frac{d\theta}{\sqrt{{}_\theta I}}. \quad . \quad . \quad . \quad (331.)$$

If we now substitute this value of $\int_0^{\frac{\pi}{2}}\frac{d\varphi}{N\sqrt{I}}$ in the equation given in (175.) for a quadrant of the logarithmic ellipse, namely,

$$\frac{2\sqrt{1-i^2}}{\sqrt{1-n}}\frac{\Sigma}{a}=\frac{[2n-i^2-n^2]}{n}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{N\sqrt{I}}+\frac{(i^2-n)}{n}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{I}}+\int_0^{\frac{\pi}{2}}d\varphi\sqrt{I},$$

since $\frac{2n-i^2-n^2}{n}=(1-i^2\sin^2\theta)-\cot^2\theta$, we shall obtain the resulting equation,

$$\frac{2\sqrt{1-i^2}}{\sqrt{(\mathrm{I}_\theta)}}\frac{\Sigma}{a}=\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{\mathrm{I}}\right]+(\mathrm{I}_\theta)\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}\right]+\mathrm{H}\left[\frac{\sqrt{(\mathrm{I}_\theta)}}{\cot\theta}-\frac{\cot\theta}{\sqrt{(\mathrm{I}_\theta)}}\right]+\text{ constant}, \quad . \quad (332.)$$

writing H for $\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}\right]\int d\theta\sqrt{(\mathrm{I}_\theta)}-\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{\mathrm{I}}\right]\int\frac{d\theta}{\sqrt{(\mathrm{I}_\theta)}}$, ... (333.)

or in the ordinary notation,

$$\mathrm{H}=\mathrm{F}_i\mathrm{E}_i(\theta)-\mathrm{E}_i\mathrm{F}_i(\theta).$$

When we require to determine the constant, we must not suppose $\theta=0$, for this would render $n=0$, and so change the nature of the curve. Neither should we be justified in making $i=0$, (as some writers do), for this would be to violate the original supposition—and all the conclusions derived from it—namely, that i is constant, and less than 1. Moreover, since $m+n-mn=i^2=0$, on this hypothesis, $m+n=mn$; or m and n would each be greater than 1, which is inconsistent with the possible values of those quantities.

We have now to determine the value of the constant. In these investigations we have all along supposed $n > m$. The least value n can have is $n=m$. Were we to suppose n to be less than m, it would be nothing more than to write m for n, since m and n are connected by the equation $m+n-mn=i^2$. Hence if m is not equal to n, one of them must be the greater, and this one we agree to call n, writing m for the lesser. To determine the constant, let us assume $n=m$.

Now $n=i^2\sin^2\theta$, and n, when equal to m, is $=1-\sqrt{1-i^2}$, $(\mathrm{I}_\theta)=1-i^2\sin^2\theta=\sqrt{1-i^2}$, $\cot^2\theta=\sqrt{1-i^2}$, and $\tan\theta=\left(\frac{1}{j}\right)^{\frac{1}{2}}$. Hence the coefficient of H in the last equation, $\frac{\sqrt{(\mathrm{I}_\theta)}}{\cot\theta}-\frac{\cot\theta}{\sqrt{(\mathrm{I}_\theta)}}$, becomes 0, since in this case $\cot\theta=\sqrt[4]{1-i^2}$; and as $n=m$, the curve is the circular logarithmic ellipse. See Art. XLIII.

The last equation now becomes

$$2\sqrt[4]{1-i^2}\frac{\Sigma}{a}=\int_0^{\frac{\pi}{2}}d\varphi\sqrt{\mathrm{I}}+\sqrt{1-i^2}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}+\text{ constant}. \quad . \quad . \quad . \quad (334.)$$

Now if we turn to (176.), we shall find this, without the constant, to be the expression for the quadrant of a circular logarithmic ellipse, or the curve in which a circular cylinder, the radius of whose base is a, intersects at an infinite distance a paraboloid indefinitely attenuated. Hence the constant is 0.

To determine the value of the above integral, when $\theta=\frac{\pi}{2}$.

In this case, as $\mathrm{H}=\mathrm{F}_i\mathrm{E}_i-\mathrm{E}_i\mathrm{F}_i$, $\mathrm{H}=0$. And as $\cot\theta=0$, and $\sqrt{\mathrm{I}_\theta}=\sqrt{1-i^2}$, the equation (332.) will assume the form

$$2\frac{\Sigma}{a}=\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{\mathrm{I}}\right]+(1-i^2)\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}\right]+\frac{0}{0}\sqrt{1-i^2}. \quad . \quad . \quad . \quad (335.)$$

How are we to interpret this expression?

To determine the value of the fraction $\frac{\mathrm{H}}{\cot\theta}$, which appears under the form of $\frac{0}{0}$ when $\theta=\frac{\pi}{2}$, we must take the first differentials of the numerator and denominator of this fraction. Now, as in (333.)

$$\mathrm{H}=\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}\right]\int d\theta\sqrt{1-i^2\sin^2\theta}-\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{\mathrm{I}}\right]\int\frac{d\theta}{\sqrt{1-i^2\sin^2\theta}}. \quad . \quad . \quad . \quad . \qquad \text{(a.)}$$

$$\text{Therefore } \frac{d\mathrm{H}}{d\theta}=\frac{\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}\right](1-i^2\sin^2\theta)-\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{\mathrm{I}}\right]}{\sqrt{1-i^2\sin^2\theta}}, \text{ and } \frac{d\cot\theta}{d\theta}=-\frac{1}{\sin^2\theta}. \quad . \quad . \qquad \text{(b.)}$$

$$\text{Hence, when } \theta=\frac{\pi}{2}, \quad \frac{\frac{d\mathrm{H}}{d\theta}}{\frac{d\cot\theta}{d\theta}}=\frac{\mathrm{H}}{\cot\theta}=\frac{\left[\left(\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}\right)(1-i^2)-\left(\int_0^{\frac{\pi}{2}}d\varphi\sqrt{\mathrm{I}}\right)\right]}{-(\sqrt{1-i^2})}. \quad . \quad . \quad . \quad . \qquad \text{(c.)}$$

Accordingly

$$\frac{\mathrm{H}}{\cot\theta}\sqrt{1-i^2\sin^2\theta}=\frac{0}{0}\sqrt{1-i^2}=-\left(\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}\right)(1-i^2)+\left(\int_0^{\frac{\pi}{2}}d\varphi\sqrt{\mathrm{I}}\right), \text{ when } \theta=\frac{\pi}{2}. \quad . \qquad (336.)$$

Substituting this value in (332.), we get $\Sigma=a\int_0^{\frac{\pi}{2}}d\varphi\sqrt{1-i^2\sin^2\varphi}$, ... (337.)

the common expression for a quadrant of a plane ellipse, whose major axis is a, and eccentricity i. As it should be, for when $\theta=\frac{\pi}{2}$, or $n=i^2$, the section of the cylinder is a plane ellipse, as shown in Case VII. p. 316. In the spherical form, the limits of θ are 0 and $\frac{\pi}{2}$, while in the paraboloidal form, the limits of θ are $\tan^{-1}\left(\frac{1}{j}\right)^{\frac{1}{2}}$ and $\frac{\pi}{2}$.

SECTION VIII.—*On Conjugate Arcs of Hyperconic Sections.*

LVIII. Conjugate arcs of hyperconic sections may be defined, as arcs whose amplitudes φ, χ, ω are connected by the equation

$$\cos\omega=\cos\varphi\cos\chi-\sin\varphi\sin\chi\sqrt{1-i^2\sin^2\omega}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (338.)$$

This is a fundamental theorem in the theory of elliptic integrals.

The angles φ, χ, ω may be called conjugate amplitudes.

When the hyperconic section is a circle, $i=0$, and $\cos\omega=\cos\varphi\cos\chi-\sin\varphi\sin\chi$, whence $\omega=\varphi+\chi$, or the conjugate amplitudes are $\varphi+\chi$, φ and χ. The development of this expression is the foundation of circular trigonometry.

On the Trigonometry of the Parabola.

When the hyperconic section is a parabola, $i=1$, and (338.) may be reduced to

$$\tan\omega=\tan\varphi\sec\chi+\tan\chi\sec\varphi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (339.)$$

If we make the imaginary transformations,

$$\tan\omega=\sqrt{-1}\sin\omega', \ \tan\varphi=\sqrt{-1}\sin\varphi', \ \tan\chi=\sqrt{-1}\sin\chi', \ \sec\varphi=\cos\varphi', \ \sec\chi=\cos\chi'. \qquad (340.)$$

The preceding formula will become, on substituting these values, and dividing by $\sqrt{-1}$,

$$\sin\omega' = \sin\varphi' \cos\chi' + \sin\chi' \cos\varphi',$$

the well-known trigonometrical expression for the sine of the sum of two circular arcs.

Hence, by the aid of imaginary transformations, we may interchangeably permute the formulæ of the trigonometry of the circle with those of the trigonometry of the parabola. In the trigonometry of the circle, $\omega=\varphi+\chi$, and in the trigonometry of the parabola ω is such a function of the angles φ and χ, as will render $\tan[(\varphi,\chi)]=\tan\varphi\sec\chi+\tan\chi\sec\varphi$. We must adopt some appropriate notation to represent this function. Let the function (φ,χ) be written $\varphi\perp\chi$, so that $\tan(\varphi\perp\chi)=\tan\varphi\sec\chi+\tan\chi\sec\varphi$. This must be taken as the *definition* of the function $\varphi\perp\chi$.

In like manner, we may represent by $\tan(\varphi\top\chi)$ the function $\tan\varphi\sec\chi-\tan\chi\sec\varphi$.

In applying the imaginary transformations, or while $\tan\varphi$ is changed into $\sqrt{-1}\sin\varphi$, $\sec\varphi$ into $\cos\varphi$, and $\cot\varphi$ into $-\sqrt{-1}\operatorname{cosec}\varphi$, $\perp$ must be changed into $+$ and $\top$ into $-$.

$\perp$ and $\top$ may be called logarithmic plus and minus. As examples of the analogy which exists between the trigonometry of the parabola and that of the circle, we give the following expressions in parallel columns; premising that the formulæ, marked by corresponding letters, may be derived singly, one from the other, by the help of the preceding imaginary transformations.

Trigonometry of the Parabola.		*Trigonometry of the Circle.*	(341.)
$\tan(\varphi\perp\chi)=\tan\varphi\sec\chi+\tan\chi\sec\varphi$.	(α.)	$\sin(\varphi+\chi)=\sin\varphi\cos\chi+\sin\chi\cos\varphi$.	(a.)
$\tan(\varphi\top\chi)=\tan\varphi\sec\chi-\tan\chi\sec\varphi$.	(β.)	$\sin(\varphi-\chi)=\sin\varphi\cos\chi-\sin\chi\cos\varphi$.	(b.)
$\sec(\varphi\,\substack{\perp\\ \top}\,\chi)=\sec\varphi\sec\chi\pm\tan\varphi\tan\chi$.	(γ.)	$\cos(\varphi\pm\chi)=\cos\varphi\cos\chi\mp\sin\varphi\sin\chi$.	(c.)
$\sin(\varphi\perp\chi)=\dfrac{\sin\varphi+\sin\chi}{1+\sin\varphi\sin\chi}$.	(δ.)	$\tan(\varphi+\chi)=\dfrac{\tan\varphi+\tan\chi}{1-\tan\varphi\tan\chi}$.	(d.)
$\sin(\varphi\top\chi)=\dfrac{\sin\varphi-\sin\chi}{1-\sin\varphi\sin\chi}$.	(ε.)	$\tan(\varphi-\chi)=\dfrac{\tan\varphi-\tan\chi}{1+\tan\varphi\tan\chi}$.	(e.)
Let $\varphi=\chi$.		Let $\varphi=\chi$.	
$\tan(\varphi\perp\varphi)=2\tan\varphi\sec\varphi$.	(η.)	$\sin2\varphi=2\sin\varphi\cos\varphi$.	(ē.)
$\sec(\varphi\perp\varphi)=\sec^2\varphi+\tan^2\varphi$.	(θ.)	$\cos2\varphi=\cos^2\varphi-\sin^2\varphi$.	(th.)
$\sin(\varphi\perp\varphi)=\dfrac{2\sin\varphi}{1+\sin^2\varphi}$.	(ι.)	$\tan2\varphi=\dfrac{2\tan\varphi}{1-\tan^2\varphi}$.	(*i.*)
$\sec\varphi=\dfrac{e^{\int\frac{d\varphi}{\cos\varphi}}+e^{-\int\frac{d\varphi}{\cos\varphi}}}{2}$, $\tan\varphi=\dfrac{e^{\int\frac{d\varphi}{\cos\varphi}}-e^{-\int\frac{d\varphi}{\cos\varphi}}}{2}$.	($\varkappa$.)	$\cos\varphi=\dfrac{e^{\varphi\sqrt{-1}}+e^{-\varphi\sqrt{-1}}}{2}$, $\sin\varphi=\dfrac{e^{\varphi\sqrt{-1}}-e^{-\varphi\sqrt{-1}}}{2\sqrt{-1}}$.	(k.)
$1+\sqrt{-1}\tan(\varphi\perp\varphi)=(\sec\varphi+\sqrt{-1}\tan\varphi)^2$.	(λ.)	$1+\sin2\varphi=(\cos\varphi+\sin\varphi)^2$.	(l.)
$\tan^2\varphi=\dfrac{\sec(\varphi\perp\varphi)-1}{2}$.	(μ.)	$\sin^2\varphi=\dfrac{1-\cos2\varphi}{2}$.	(m.)
Let the amplitudes be $\varphi\perp\chi$ and $\varphi\top\chi$.		Let the amplitudes be $\varphi+\chi$ and $\varphi-\chi$.	
$\tan(\varphi\perp\chi)\tan(\varphi\top\chi)=\tan^2\varphi-\tan^2\chi$.	(ν.)	$\sin(\varphi+\chi)\sin(\varphi-\chi)=\sin^2\varphi-\sin^2\chi$.	(n.)

Since $\sec(\varphi\perp\varphi)=\sec^2\varphi+\tan^2\varphi$, and $\tan(\varphi\perp\varphi)=2\tan\varphi\sec\varphi$,

$\sec(\varphi\perp\varphi)+\tan(\varphi\perp\varphi)=(\sec\varphi+\tan\varphi)^2$.

Again, as $\sec(\varphi\perp\varphi\perp\varphi)=\sec(\varphi\perp\varphi)\sec\varphi+\tan(\varphi\perp\varphi)\tan\varphi$,

and $\tan(\varphi\perp\varphi\perp\varphi)=\tan(\varphi\perp\varphi)\sec\varphi+\sec(\varphi\perp\varphi)\tan\varphi$,

it follows that $\sec(\varphi\perp\varphi\perp\varphi)+\tan(\varphi\perp\varphi\perp\varphi)=(\sec\varphi+\tan\varphi)^3$,

and so on to any number of angles. Hence

$$\sec(\varphi\perp\varphi\perp\varphi\ldots \text{ to } n\varphi)+\tan(\varphi\perp\varphi\perp\varphi\ldots. \text{ to } n\varphi)=(\sec\varphi+\tan\varphi)^n. \quad . \quad . \quad . \quad (342.)$$

Introduce into the last expression the imaginary transformation, $\tan\varphi=\sqrt{-1}\sin\varphi$, and we get DEMOIVRE's imaginary theorem for the circle,

$$\cos n\varphi+\sqrt{-1}\sin n\varphi=(\cos\varphi+\sqrt{-1}\sin\varphi)^n.$$

Let $\bar{\omega}$ be conjugate to ψ and ω, while ω, as before, is conjugate to φ and χ. Then we shall have $\tan\bar{\omega}=\tan(\varphi\perp\chi\perp\psi)$, or

$$\tan(\varphi\perp\chi\perp\psi)=\tan\varphi\sec\chi\sec\psi+\tan\chi\sec\psi\sec\varphi+\tan\psi\sec\varphi\sec\chi+\tan\varphi\tan\chi\tan\psi, \quad . \quad . \quad . \quad (\varpi.)$$

$$\sec(\varphi\perp\chi\perp\psi)=\sec\varphi\sec\chi\sec\psi+\sec\varphi\tan\chi\tan\psi+\sec\chi\tan\psi\tan\varphi+\sec\psi\tan\varphi\tan\chi, \quad . \quad . \quad . \quad (\rho.)$$

and $$\sin(\varphi\perp\chi\perp\psi)=\frac{\sin\varphi+\sin\chi+\sin\psi+\sin\varphi\sin\chi\sin\psi}{1+\sin\chi\sin\psi+\sin\psi\sin\varphi+\sin\varphi\sin\chi}; \quad . \quad . \quad . \quad (\sigma.)$$

whence, in the trigonometry of the circle,

$$\sin(\varphi+\chi+\psi)=\sin\varphi\cos\chi\cos\psi+\sin\chi\cos\psi\cos\varphi+\sin\psi\cos\varphi\cos\chi-\sin\varphi\sin\chi\sin\psi, \quad . \quad . \quad . \quad (\text{p.})$$

$$\cos(\varphi+\chi+\psi)=\cos\varphi\cos\chi\cos\psi-\cos\varphi\sin\chi\sin\psi-\cos\chi\sin\psi\sin\varphi-\cos\psi\sin\varphi\sin\chi, \quad . \quad . \quad . \quad (\text{r.})$$

$$\tan(\varphi+\chi+\psi)=\frac{\tan\varphi+\tan\chi+\tan\psi-\tan\varphi\tan\chi\tan\psi}{1-\tan\chi\tan\psi-\tan\psi\tan\varphi-\tan\varphi\tan\chi} \quad . \quad . \quad . \quad (\text{s.})$$

LIX. Let $(k.\omega)$, $(k.\varphi)$, $(k.\chi)$ denote three parabolic arcs measured from the vertex of the parabola whose parameter is k.

The normal angles of these arcs are ω, φ, and χ; ω, φ and χ being conjugate amplitudes. Then

$$2(k.\varphi)=k\tan\varphi\sec\varphi+k\int\frac{d\varphi}{\cos\varphi},\quad 2(k.\chi)=k\tan\chi\sec\chi+k\int\frac{d\chi}{\cos\chi},\quad 2(k.\omega)=k\tan\omega\sec\omega+k\int\frac{d\omega}{\cos\omega};$$

whence, since $\int\frac{d\omega}{\cos\omega}-\int\frac{d\varphi}{\cos\varphi}-\int\frac{d\chi}{\cos\chi}=0$, because ω, φ, and χ are conjugate amplitudes,

$$(k.\omega)-(k.\varphi)-(k.\chi)=k\tan\omega\tan\varphi\tan\chi. \quad . \quad . \quad . \quad (343.)$$

Let y, y', y'' be the ordinates of the arcs $(k.\varphi)$, $(k.\chi)$, and $(k.\omega)$. Then $y=k\tan\varphi$, $y'=k\tan\chi$, $y''=k\tan\omega$, and the last expression becomes

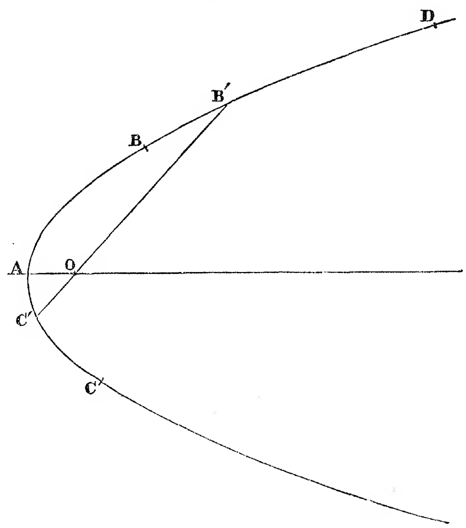
$$(k.\omega)-(k.\varphi)-(k.\chi)=\frac{yy'y''}{k^2}. \quad . \quad . \quad . \quad (344.)$$

If we call an arc measured from the vertex of a parabola an *apsidal* arc, to distinguish it from an arc taken anywhere along the parabola, the preceding theorem

will enable us to express an arc of a parabola, taken anywhere along the curve, as the sum or difference of an apsidal arc and a right line.

Thus let ACD be a parabola, O its focus and A its vertex. Let $AB=(k.\varphi)$, $AC=(k.\chi)$, $AD=(k.\omega)$ and $\frac{yy'y''}{k^2}=h$. Then (343.) shows that the parabolic arc $(AC+AB)=$apsidal arc $AD-h$; and the parabolic arc $(AD-AB)=BD=$apsidal arc $AC+h$. When the arcs AC′, AB′ together constitute a focal arc, or an arc whose cord passes through the focus, $\varphi+\chi=\frac{\pi}{2}$, and h is the ordinate of the conjugate arc AD. Hence we derive this theorem,

Fig. 21.



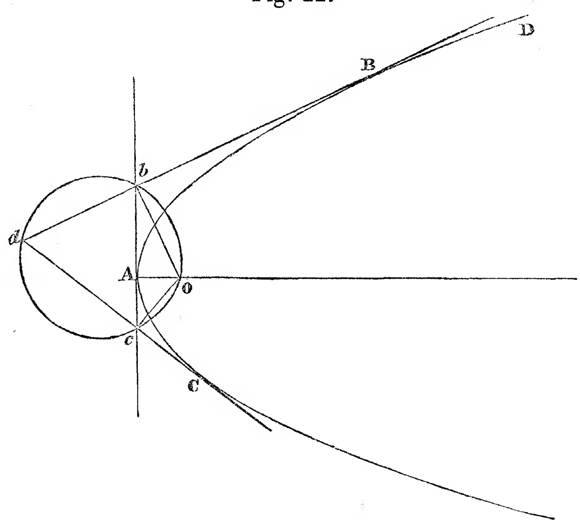
Any focal arc of a parabola is equal to the difference between the conjugate apsidal arc and its ordinate.

The relation between the amplitudes φ and ω in this case is $\sin 2\varphi=\frac{2\cos\omega}{1-\cos\omega}$. Thus when the focal cord makes an angle of 30° with the axis, we get $\cos\omega=\frac{1}{5}$, or $y=5k$. Here therefore the ordinate of the conjugate arc is five times the semiparameter.

LX. We may, in all cases, represent by a simple geometrical construction, the ordinates of the conjugate parabolic arcs, whose amplitudes are φ, χ and ω.

Let ABC be a parabola whose focus is O, and whose vertex is A. Let $AO=g=\frac{k}{2}$; moreover let AB be the arc whose amplitude is φ, and AC the arc whose amplitude is χ. At the points A, B, C draw tangents to the parabola, they will form a triangle circumscribing the parabola, whose sides represent the semi-ordinates of the conjugate arcs, AB, AC, AD.

Fig. 22.



We know that the circle, circumscribing this triangle, passes through the focus of the parabola.

Now $\quad Ab=g\tan\varphi,\ Ac=g\tan\chi,\ bd=g\tan\varphi\sec\chi,\ cd=g\tan\chi\sec\varphi;$

hence $\quad bd+cd=g(\tan\varphi\sec\chi+\tan\chi\sec\varphi)$, therefore $g\tan\omega=bd+cd$.

When AB, AC together constitute a focal arc, the angle bdc is a right angle.

The diameter of this circle is $g\sec\varphi\sec\chi$.

The demonstration of these properties follows obviously from the figure.

LXI. In the trigonometry of the circle, we find the formula

$$\vartheta=\tan\vartheta-\frac{\tan^3\vartheta}{3}+\frac{\tan^5\vartheta}{5}-\frac{\tan^7\vartheta}{7}+\&c.; \quad \text{(a.)}$$

and if we develop, by common division, the expression

$\frac{1}{\cos\theta}=\frac{\cos\theta}{1-\sin^2\theta}=\cos\theta(1+\sin^2\theta+\sin^4\theta+\sin^6\theta+\ldots\&c.)$ and integrate,

$$\int\frac{d\theta}{\cos\theta}=\sin\theta+\frac{\sin^3\theta}{3}+\frac{\sin^5\theta}{5}+\frac{\sin^7\theta}{7}+\ldots\&c. \quad \text{(b.)}$$

If we now inquire, what, in the circle, is the arc which differs from its protangent, by the distance between the vertex and its focus; or, as the protangent is 0 in the circle, and the focus is the centre; the question may be changed into what is the trigonometrical tangent of the arc of a circle equal to the radius. This question is answered by putting 1 for ϑ in (a.), and reverting the series

$$1=\tan(1)-\frac{\tan^3(1)}{3}+\frac{\tan^5(1)}{5}-\frac{\tan^7(1)}{7}+\&c.;$$

we should get, in functions of the numbers of BERNOUILLI, the value of $\tan(1)$, as is shown in most treatises on trigonometry.

Let us now make a like inquiry in the case of the parabola, and ask what is the value of the amplitude which will give the difference, between the arc of the parabola and its protangent, equal to the distance between the focus and the vertex of the parabola. Now if θ be this angle, we must have $(k.\theta)-g\sec\theta\tan\theta=g$. But in general, $(k.\theta)-g\sec\theta\tan\theta=g\int\frac{d\theta}{\cos\theta}$. Hence we must have, in this case, $\int\frac{d\theta}{\cos\theta}=1$. If we now revert the series (b.), putting 1 for $\int\frac{d\theta}{\cos\theta}$, we shall get from this particular value of the series,

$$1=\sin\theta+\frac{\sin^3\theta}{3}+\frac{\sin^5\theta}{5}+\frac{\sin^7\theta}{7}+\&c.,$$

an arithmetical value for $\sin\theta$. This will be found to be, $\sin\theta=\frac{e^1-e^{-1}}{e^1+e^{-1}}$, e being the base of the Napierian logarithms. Hence $\sec\theta+\tan\theta=e$, or if we write ϵ for this particular value of θ to distinguish it from every other, and call it the *angle of the base*,

$$\sec\epsilon+\tan\epsilon=e. \quad (345.)$$

We are thus (for the first time it is believed) put in possession of the geometrical origin of that quantity, so familiarly known to mathematicians, the Napierian base. From the above equations we may derive

$$\sec\epsilon=\frac{e^1+e^{-1}}{2},\ \tan\epsilon=\frac{e^1-e^{-1}}{2}, \quad (346.)$$

or $\tan\epsilon=1{\cdot}175203015$, whence $\epsilon={\cdot}8657606$, or $\epsilon=49^\circ.\ 36'.\ 15''$.

The corresponding arc of the parabola will be $(k.\theta)=k\left[1+\frac{2^1}{123}+\frac{2^3}{12345}+\&c.\right]$. (347.)

If we assume the theory of logarithms as known, we may at once arrive at this value, for in general

$$\int\frac{d\theta}{\cos\theta}=\log(\sec\theta+\tan\theta);$$

and as this is to be 1, we must have $\sec\theta+\tan\theta=e$, as before.

LXII. If we now extend this inquiry, and ask, what is the magnitude of the amplitude of the arc of the parabola which shall render the difference between the parabolic arc and its protangent equal to n times the distance between the focus and the vertex; we shall have, as before, by the terms of the question,

$$(k.\theta)-g\sec\theta\tan\theta=ng. \qquad (348.)$$

But in general

$$(k.\theta)-g\sec\theta\tan\theta=g\int\frac{d\theta}{\cos\theta};$$

hence we must have

$$n=\int\frac{d\theta}{\cos\theta}=\log(\sec\theta+\tan\theta),$$

or

$$\sec\theta+\tan\theta=e^n. \qquad (349.)$$

Now we may solve this equation in two ways; either by making n a given number, and then determine the value of $\sec\theta+\tan\theta$, which may be called the *base*. Or we may assign an arbitrary value to $\sec\theta+\tan\theta$, and then derive the value of n. Taking the latter course, let, for example,

$$\sec\theta+\tan\theta=10. \quad \text{Then } n=\log 10,$$

or $\frac{1}{n}$ is the modulus of the second system of logarithms. Hence, if we assume any number of systems of logarithms on the same parabola, and take their bases

$$g(\sec\theta+\tan\theta), \quad g(\sec\theta'+\tan\theta'), \quad g(\sec\theta''+\tan\theta''), \ldots \&c.,$$

the moduli of these successive systems will be the ratios of half the semiparameter to the successive differences between the base parabolic arcs and their protangents.

In the Napierian system, g the distance from the focus to the vertex of the parabola, is taken as 1. The difference between the parabolic arc and its protangent, when equal to g, gives $g(\sec\theta+\tan\theta)=eg$. In the decimal system $g(\sec\theta_{,}+\tan\theta_{,})=10g$, and the difference between the corresponding parabolic arc and its protangent being ng, if we make this difference ng equal to the *arithmetical unit*, we shall have $ng=1$, or $g=\frac{1}{n}=$modulus of the system. Hence in every system of logarithms whatever, g the distance between the focus and the vertex of the parabola, is the modulus of the system. Every system of logarithms may be derived from the same parabola, but the Napierian system, in which the focal distance of the vertex is itself taken as the unit, may justly be taken as the *natural* system. In the same way we may consider that to be the *natural* system of circular trigonometry, in which the radius is taken as the unit. The modulus, in the trigonometry of the parabola, corresponds with the radius in the trigonometry of the circle. But while in the trigonometry of the parabola the base is real, in the circle it is imaginary. In the parabola, the angle of the base is given by the equation $\sec\theta+\tan\theta=e$. In the circle $\cos\vartheta+\sqrt{-1}\sin\vartheta=e^{\vartheta\sqrt{-1}}$, and

making $\vartheta=1$, we get $\cos(1)+\sqrt{-1}\sin(1)=e^{\sqrt{-1}}$. Hence while e^1 is the *parabolic* base, $e^{\sqrt{-1}}$ is the *circular* base. Or as $[\sec\epsilon+\tan\epsilon]$ is the Napierian base, $[\cos(1)+\sqrt{-1}\sin(1)]$ is the *circular* or imaginary base. Thus

$$[\cos(1)+\sqrt{-1}\sin(1)]^{\vartheta}=\cos\vartheta+\sqrt{-1}\sin\vartheta.$$

Hence, speaking more precisely, imaginary numbers have real logarithms, but an imaginary base. We may always pass from the real logarithms of the parabola, to the imaginary logarithms of the circle, by changing $\tan\theta$ into $\sqrt{-1}\sin\vartheta$, $\sec\theta$ into $\cos\vartheta$, and e^1 into $e^{\sqrt{-1}}$.

As in the parabola the angle θ is non-periodic, its limit being $\pm\frac{\pi}{2}$, while in the circle ϑ has no limit, it follows that while a number can have only one real or *parabolic* logarithm, it may have innumerable imaginary or *circular* logarithms.

In the parabola we thus can show the geometrical origin of the magnitudes known as the base and the modulus. We might too form systems of circular trigonometry analogous to different systems of logarithms. We might refer the arc of a circle not to the radius, but to some other arbitrary fixed line, the diameter or any other suppose. Let the circumference be referred to the diameter, then π will signify a whole circumference instead of a semicircle, and $\frac{\pi}{4}$ will represent a right angle. Having on this system, or any similar one, found the lengths of the arcs which correspond to certain functions, such as given sines or tangents, we should multiply the results by some fixed number, which we might call a modulus (2 in this example), to reduce them to the standard system; but such systems would obviously be useless.

If ϵ be the angle which gives the difference between the parabolic arc and its protangent equal to $g=\frac{k}{2}$; $(\epsilon\perp\epsilon)$ is the angle which will give this difference equal to $2g$, $(\epsilon\perp\epsilon\perp\epsilon)$ is the angle which will give this difference equal to $3g$, and so on to any number of angles. Hence, in the circle, if ϑ be the angle which gives the circular arc equal to the radius, 2ϑ is the angle which will give an arc equal to twice the radius, and so on for any number of angles. This is of course self-evident in the case of the circle, but it is instructive to point out the complete analogy which holds in the trigonometries of the circle and of the parabola.

LXIII. The geometrical origin of the exponential theorem may thus be shown.

Assume two known logarithmic bases $(\sec\alpha+\tan\alpha)$, and $(\sec\beta+\tan\beta)$, and let us investigate the ratio of the differences of the corresponding parabolic arcs and their protangents.

Let $\sec\epsilon+\tan\epsilon$ be the Napierian base, and let one difference be xg and the other yg. The ratio of these differences is therefore $\frac{y}{x}=z$, if we make $y=xz$. Hence

$$\sec\alpha+\tan\alpha=(\sec\epsilon+\tan\epsilon)^x=e^x, \text{ and } (\sec\beta+\tan\beta)=e^y. \text{ Therefore}$$

$$(\sec\alpha+\tan\alpha)^y=e^{x+y}=(\sec\beta+\tan\beta)^x.$$

Or, as $y=xz$, $$(\sec\alpha+\tan\alpha)^z=\sec\beta+\tan\beta.$$

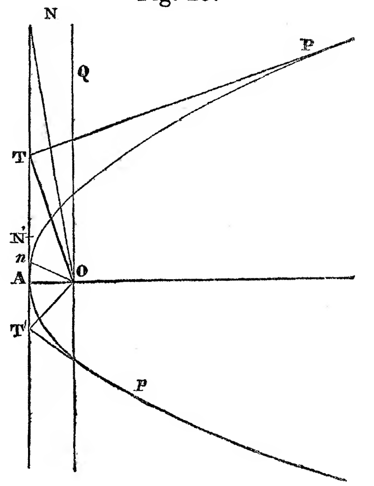
Let A be the first base, and B the second. Then $B=A^y$. This is the exponential theorem.

Let A be the Napierian base, then $x=1$, and $A=e$. Hence $B=e^y$.

LXIV. Given the number to find its logarithm, may be exhibited by the following geometrical construction.

Let OAP be a parabola. Through the focus O draw the perpendicular OQ to the axis AO. Through A let a tangent of indefinite length be drawn. On this tangent take the line AN to represent the given number. Join NO, and make the angle NOT always equal to the angle NOQ. Draw TP at right angles to TO. This line will touch the parabola in the point P, and the arc of the parabola AP—PT will be the logarithm of AN.

Fig. 23.



When $AN'=AO=$the unit g, the angle N′OQ is equal to half a right angle. Hence the point T in this case will coincide with A. The parabolic arc therefore vanishes, or the logarithm of 1 is 0. When $\sec\theta+\tan\theta=1$, $\theta=0$.

When the number is less than 1, the point N will fall below N′ in the position n. Hence nOQ is greater than half a right angle. Therefore T will fall *below* the axis in the point T′; and if we draw through T′ a tangent T′p, it will give the *negative* arc of the parabola T′p, corresponding to the number An. Fractional numbers, or numbers between $+1$ and 0, must therefore be represented by the expression $g(\sec\theta-\tan\theta)$, since $\tan\theta$ changes its sign.

When the number is 0, n coincides with A, and the angle NOQ in this case is a right angle. Therefore the point T′ will be the intersection of AT′ and OQ. Hence T′ is at an infinite distance below the axis, and therefore the logarithm of $+0$ is $-\infty$.

Hence negative numbers have no logarithms, at least no real ones; and imaginary ones can only be educed by the transformation so often referred to, and this leads us to seek them among the properties of the circle. For as θ always lies between 0 and a right angle, or between 0 and the half of $\pm\varpi$, $\sec\theta\pm\tan\theta$ is *always* positive; hence *negative* numbers can have no real or *parabolic* logarithms, but they may have imaginary or *circular* logarithms; for in the expression $\log(\cos\vartheta+\sqrt{-1}\sin\vartheta)=\vartheta\sqrt{-1}$, we may make $\vartheta=(2n+1)\pi$, and we shall get $\log(-1)=(2n+1)\pi\sqrt{-1}$.

Hence also, as the length of the parabolic arc TP, without reference to the sign, depends solely on the amplitude θ, it follows that the logarithm of $\sec\theta-\tan\theta$ is equal to the logarithm of $\sec\theta+\tan\theta$. As $(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)=1$, we may hence infer, that the logarithm of any number is equal to the logarithm of its reciprocal, with the sign changed.

When θ is very large, $\sec\theta+\tan\theta=2\tan\theta$, nearly. Hence if we represent a large number by an ordinate of a parabola whose focal distance to the vertex is 1, the difference between the corresponding arc and its protangent will represent its logarithm.

Along the tangent to the vertex of the parabola, as in the preceding figure, draw,

measured from the vertex, a series of lines in geometrical progression,

$$g(\sec\theta+\tan\theta),\quad g(\sec\theta+\tan\theta)^2,\quad g(\sec\theta+\tan\theta)^3 \ldots\ldots g(\sec\theta+\tan\theta)^n.$$

Join N, the general representative of the extremities of these right lines, with the focus O. Erect the perpendicular OQ, and make the angle NOT *always* equal to the angle NOQ. The line OT will be $=g\sec\theta$, the line $OT_{/}=g\sec(\theta\perp\theta)$, the line $OT_{//}=g\sec(\theta\perp\theta\perp\theta)$, &c., and we shall likewise have

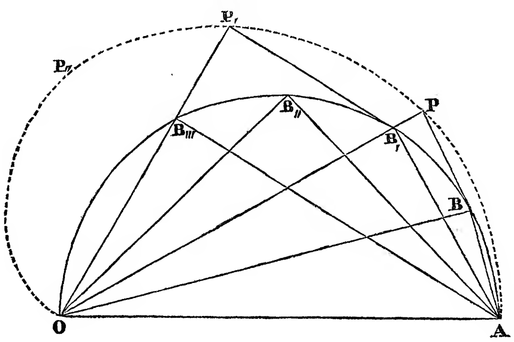
$$AT=g\tan\theta,\quad AT_{/}=g\tan(\theta\perp\theta),\quad AT_{//}=g\tan(\theta\perp\theta\perp\theta),\ \&c.$$

This follows immediately from (342.); for any integral power of $(\sec\theta+\tan\theta)$ may be exhibited as a linear function of $\sec\Theta+\tan\Theta$, if $\Theta=\theta\perp\theta\perp\theta\ldots\&c.$,

since $\sec(\theta\perp\theta\perp\theta\perp\theta\ \&c.\text{ to } n\theta)+\tan(\theta\perp\theta\perp\theta\perp\theta\ \&c.\text{ to } n\theta)=(\sec\theta+\tan\theta)^n.$

Hence the parabola enables us to give a graphical construction for the angle $(\theta\perp\theta\perp\&c.)$ as the circle does for the angle $n\theta$.

Fig. 24.



The analogous theorem in the circle may be developed as follows:—In the circle OBA, (fig. 24) take the arcs $AB=BB_{/}=B_{/}B_{//}=B_{//}B_{///}\ldots\&c.=2\vartheta$. Let the diameter be G. Then $OB=G\cos\vartheta, OB_{/}=G\cos2\vartheta, OB_{//}=G\cos3\vartheta\ldots$ &c. and $AB=G\sin\vartheta$, $AB_{/}=G\sin2\vartheta, AB_{//}=G\sin3\vartheta\ldots$ &c.

Now as the lines in the second group are always at right angles to those in the first, and as such a change is denoted by the symbol $\sqrt{-1}$, we get

$$OB+BA=G(\cos\vartheta+\sqrt{-1}\sin\vartheta),$$

$$OB_{/}+B_{/}A=G(\cos2\vartheta+\sqrt{-1}\sin2\vartheta)=G(\cos\vartheta+\sqrt{-1}\sin\vartheta)^2;$$

$$OB_{//}+B_{//}A=G(\cos3\vartheta+\sqrt{-1}\sin3\vartheta)=G(\cos\vartheta+\sqrt{-1}\sin\vartheta)^3\&c.$$

LXV. The known theorem, that a parabola is the reciprocal polar of a circle, whose circumference passes through the focus, suggests a transformation, which will exhibit a much closer analogy between the formulæ for the rectification of the parabola and the circle, than when the centre of the latter curve is taken as the origin.

Let OBA be a semicircle, let the origin be placed at O, let the angle $AOB=\vartheta$, and let G, as before, be the diameter of the circle. Through B draw the tangent BP; let fall on this tangent the perpendicular $OP=p$, and let BP the protangent be equal to t.

Now as $p=G\cos^2\vartheta$, and $t=G\sin\vartheta\cos\vartheta$, as also the angle $AOP=2\vartheta$, if we apply to the circle the formula for rectification in (33.), we shall have the arc

$$AB=s=2G\int\cos^2\vartheta d\vartheta-G\sin\vartheta\cos\vartheta.$$

Make the imaginary transformations $\cos\vartheta=\sec\theta$, and $\sin\vartheta=\sqrt{-1}\tan\theta$, and we shall have

$$\frac{s}{G\sqrt{-1}}=2\int\frac{d\theta}{\cos^3\theta}-\sec\theta\tan\theta.$$

The expression for an arc of a parabola, diminished by its protangent.

The protangent to the circle, which is exhibited in this formula, disappears in the actual process of integration; while in the parabola, the protangent which is involved in the differential, is evolved by the process of integration.

As in the parabola, the perpendicular, from the focus on the tangent, bisects the angle between the radius vector and the axis of the curve; so in the circle, the radius vector OB drawn from the extremity of the diameter, bisects the angle between the perpendicular OP and the diameter OA.

There are some curious analogies between the parabola and the circle, considered under this point of view.

In the parabola, the points T, $T_{\prime}$, $T_{\prime\prime}$, which divide the lines

$$g\,(\sec\theta+\tan\theta),\quad g\,[\sec(\theta\perp\theta)+\tan(\theta\perp\theta)],\ \&c.$$

into their component parts, are upon tangents to the parabola. The corresponding points B, $B_{\prime}$, $B_{\prime\prime}$ in the circle, are on the circumference of the circle.

In the parabola the extremities of the lines $g\,(\sec\theta+\tan\theta)$ are on a right line AN; in the circle, the extremities of the bent lines $G\,(\cos\vartheta+\sqrt{-1}\,\sin\vartheta)$ are all in the point A.

The locus of the point T, the intersections of the tangents to the parabola with the perpendiculars from the focus, is a right line; or in other words, while one end of a protangent rests on the parabola, the other end rests on a right line. So in the circle, while one end of the protangent rests on the circle, the other end rests on a *cardioide*, whose diameter is equal to that of the circle, and whose cusp is at O. $OPP_{\prime}A$ is the cardioide.

The length of the tangent AT to any point T is $g\tan\theta$. The length of the cardioide is $2G\sin\vartheta$.

It is singular that the imaginary formulæ in trigonometry have long been discovered, while the corresponding real expressions have escaped notice. Indeed, it was long ago observed by LAMBERT, and by other geometers—the remark has been repeated in almost every treatise on the subject since—that the ordinates of an equilateral hyperbola might be expressed by real exponentials, whose exponents are sectors of the hyperbola; but the analogy, being illusory, never led to any useful results. And the analogy was illusory from this, that it so *happens* the length and area of a circle are expressed by the *same* function, while the area of an equilateral hyperbola is a function of an arc of a parabola. The true analogue of the circle is the parabola.

LXVI. Let $\bar{\omega}$ be the conjugate amplitude of ω and ψ, while ω is the conjugate amplitude, as before, of ϕ and χ.

Then as $\displaystyle\int\frac{d\bar{\omega}}{\cos\bar{\omega}}=\int\frac{d\omega}{\cos\omega}+\int\frac{d\psi}{\cos\psi}$, and $\displaystyle\int\frac{d\omega}{\cos\omega}=\int\frac{d\phi}{\cos\phi}+\int\frac{d\chi}{\cos\chi}$,
we shall have

$$\int\frac{d\bar{\omega}}{\cos\bar{\omega}}=\int\frac{d\phi}{\cos\phi}+\int\frac{d\chi}{\cos\chi}+\int\frac{d\psi}{\cos\psi};$$

and if $(k.\bar{\omega})$, $(k.\varphi)$, $(k.\chi)$ and $(k.\psi)$ are four corresponding parabolic arcs,

$$(k.\bar{\omega})-(k.\varphi)-(k.\chi)-(k.\psi)=k\tan(\varphi\perp\chi)\tan(\varphi\perp\psi)\tan(\chi\perp\psi), \quad . \quad . \quad (350.)$$

which gives a simple relation between four conjugate parabolic arcs.

Let, in the preceding formula, $\varphi=\chi=\psi$, and we shall have

$$(k.\bar{\omega})-3(k.\varphi)=k\tan^3(\varphi\perp\varphi)=8k\tan^3\varphi\sec^3\varphi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (351.)$$

We are thus enabled to assign the difference between an arc of a parabola and three times another arc, $\bar{\omega}=(\varphi\perp\varphi\perp\varphi)$.

If in (ϖ) (341.) we make $\varphi=\chi=\psi$, $\tan\bar{\omega}=4\tan^3\varphi+\tan\varphi$.

Introduce into this expression, the imaginary transformation $\tan\varphi=\sqrt{-1}\sin\theta$, and we shall get $\sin3\theta=-4\sin^3\theta+\sin\theta$, which is the known formula for the trisection of a circular arc. (351.) may therefore be taken as the formula which gives the trisection of an arc of a parabola.

When there are five parabolic arcs, whose normal angles φ, χ, ψ, υ, Ω are related as above, namely,

$$\omega=\varphi\perp\chi,\quad \bar{\omega}=\omega\perp\psi=\varphi\perp\chi\perp\psi,\quad \Omega=\varphi\perp\chi\perp\psi\perp\upsilon,$$

we get the following relation,

$$(k.\Omega)-(k.\varphi)-(k.\chi)-(k.\psi)-(k.\upsilon)=k\tan(\varphi\perp\chi\perp\upsilon)\tan(\chi\perp\psi\perp\upsilon)\tan(\psi\perp\varphi\perp\upsilon), (352.)$$

a formula which connects five parabolic arcs, whose amplitudes are derived by the given law.

We might pursue this subject very much further; but enough has been done to show the analogy which exists between the trigonometry of the circle and that of the parabola. As the calculus of angular magnitude has always been referred to the circle as its type, so the calculus of logarithms may, in precisely the same way, be referred to the parabola as its type.

The obscurities, which hitherto have hung over the geometrical theory of logarithms, have it is hoped been now removed. It is possible to represent logarithms, as elliptic integrals usually have been represented, by curves devised to exhibit some special property only; and accordingly, such curves, while they place before us the properties they have been constructed to represent, fail generally to carry us any further. The close analogies which connect the theory of logarithms with the properties of the circle will no longer appear inexplicable*.

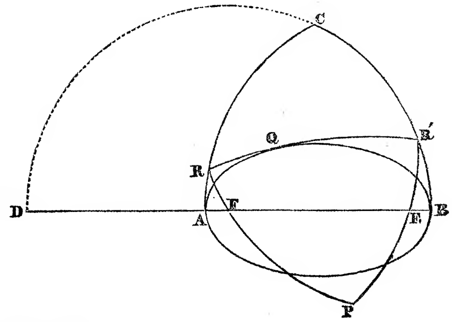
* The views above developed, on the trigonometry of the parabola, throw much light on a controversy long carried on between LEIBNITZ and J. BERNOULLI on the subject of the logarithms of negative numbers. LEIBNITZ insisted they were imaginary, while BERNOULLI argued they were real, and the same as the logarithms of equal positive numbers. EULER espoused the side of the former, while D'ALEMBERT coincided with the views of BERNOULLI. Indeed, if we derive the theory of logarithms from the properties of the hyperbola (as geometers always have done), it will not be easy satisfactorily to answer the argument of BERNOULLI—that as an hyperbolic area represents the logarithm of a positive number, denoted by the positive abscissa $+x$, so a negative number, according to conventional usage, being represented by the negative abscissa $-x$, the corresponding hyperbolic area should denote its logarithm also. All this obscurity is cleared up by the theory developed in the text, which completely establishes the correctness of the views of LEIBNITZ and EULER.

On Conjugate Arcs of a Spherical Parabola.

LXVII. The well-known relations between elliptic integrals of the first order, whose amplitudes are conjugate, develope some very elegant geometrical theorems.

Thus in fig. (25.), since the arc $AQ=j\int\frac{d\phi}{\sqrt{I_\phi}}+QR$, and the arc $BQ=j\int\frac{d\chi}{\sqrt{I_\chi}}+QR'$, the arcs $AQ+BQ=j\left[\int\frac{d\phi}{\sqrt{I_\phi}}+\int\frac{d\chi}{\sqrt{I_\chi}}\right]+QR+QR'$. . . . . (a.)

Fig. 25.



Now $AQ+BQ=$ two quadrants of the spherical parabola, and $QR+QR'=\frac{\pi}{2}$, whence half the circumference, or $AQB=j\left[\int\frac{d\phi}{\sqrt{I_\phi}}+\int\frac{d\chi}{\sqrt{I_\chi}}\right]+\frac{\pi}{2}$.

In XXII. it has been shown that the complete integral represents the semicircumference, whence

$$AQB=j\int_0^{\frac{\pi}{2}}\frac{d\omega}{\sqrt{I_\omega}}+\frac{\pi}{2}. \quad . \quad . \quad . \quad . \quad . \quad (b.)$$

Comparing these equations (a.) and (b.) together, we get

$$\int_0^{\frac{\pi}{2}}\frac{d\omega}{\sqrt{I_\omega}}=\int\frac{d\phi}{\sqrt{I_\phi}}+\int\frac{d\chi}{\sqrt{I_\chi}}.$$

Now as the triangle RR'P is a quadrantal right-angled triangle, the relation between the angles AFR, BER', or ϕ and χ, is easily discovered. Since FPE is a spherical triangle right-angled at P, and $FE=2\varepsilon=\frac{\pi}{2}-\gamma$, we get $j\tan\phi\tan\chi=1$. When $AQ=BQ$, $\phi=\chi$, and $\tan\phi=\frac{1}{\sqrt{j}}$.

The locus of the point P is a spherical ellipse, supplemental to the former, having the extremities of its principal minor arc, in the foci F, E of the former.

LXVIII. Let $\sigma, \sigma_{,}, \sigma_{,,}$ be three arcs of a spherical parabola, corresponding to the conjugate amplitudes ϕ, χ, ω. Then successively substituting these amplitudes in (58.), the resulting equation becomes

$$\sigma+\sigma_{,}-\sigma_{,,}=j\left[\int\frac{d\phi}{\sqrt{I_\phi}}+\int\frac{d\chi}{\sqrt{I_\chi}}-\int\frac{d\omega}{\sqrt{I_\omega}}\right]+\tau+\tau_{,}-\tau_{,,}.$$

But as the amplitudes ϕ, χ, ω are conjugate, the sum of these integrals of the first order is 0, whence

$$\sigma+\sigma_{,}-\sigma_{,,}=\tau+\tau_{,}-\tau_{,,}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (353.)$$

Or, when the amplitudes of three arcs in the spherical parabola are conjugate amplitudes, the sum of the arcs is equal to the sum of the protangents. We use the word sum in its algebraic sense.

On Conjugate Arcs of a Spherical Ellipse.

LXIX. If, in (42.), we substitute successively φ, χ, ω, and add the resulting equations, we shall have

$$\sigma+\sigma_{,}-\sigma_{,,}=\left(\frac{1+n}{n}\right)\sqrt{mn}\left[\int\frac{d\varphi}{N_{\varphi}\sqrt{I_{\varphi}}}+\int\frac{d\chi}{N_{\chi}\sqrt{I_{\chi}}}-\int\frac{d\omega}{N_{\omega}\sqrt{I_{\omega}}}\right]$$

$$-\frac{i^2}{\sqrt{mn}}\left[\int\frac{d\varphi}{\sqrt{I_{\varphi}}}+\int\frac{d\chi}{\sqrt{I_{\chi}}}-\int\frac{d\omega}{\sqrt{I_{\omega}}}\right]-\tau-\tau_{,}+\tau_{,,}. \qquad (354.)$$

Now the conjugate relation between φ, χ and ω renders the sum of the integrals of the first order$=0$, and the sum of the integrals of the third order equal to a circular arc Θ, which is given by the equation

$$\tan\Theta=\frac{\sqrt{mn}\sin\varphi\sin\chi\sin\omega}{1-\frac{n}{1+n}\cos\varphi\cos\chi\cos\omega}. \qquad (355.)$$

Hence $$\sigma+\sigma_{,}-\sigma_{,,}=\Theta-\tau-\tau_{,}+\tau_{,,}. \qquad (356.)$$

Or, *when the amplitudes are conjugate, the sum of three arcs of a spherical ellipse may be expressed as the sum of four circular arcs.*

When one of the amplitudes ω is a right angle, $\sigma_{,,}$ becomes a quadrant of the spherical ellipse$=\breve{\sigma}$. $\tau_{,,}=0$, and $\Theta=\tau=\tau_{,}$, as we shall show presently, whence

$$(\sigma-\sigma_{,})-\breve{\sigma}=\tau, \text{ which agrees with (52.).}$$

Or the difference between two arcs of a spherical ellipse, measured from the vertices of the curve, may be expressed by a circular arc. In (45.) we found

$$\tan\tau=\frac{\sqrt{mn}\sin\varphi\cos\varphi}{\sqrt{1-i^2\sin^2\varphi}}, \quad \tan\tau_{,}=\frac{\sqrt{mn}\sin\chi\cos\chi}{\sqrt{1-i^2\sin^2\chi}}.$$

Now when $\omega=\frac{\pi}{2}$, (338.) gives $\sin\chi=\frac{\cos\varphi}{\sqrt{1-i^2\sin^2\varphi}}$, $\sin\varphi=\frac{\cos\chi}{\sqrt{1-i^2\sin^2\chi}}$,

whence $$\sqrt{mn}\sin\varphi\sin\chi=\frac{\sqrt{mn}\sin\varphi\cos\varphi}{\sqrt{1-i^2\sin^2\varphi}}=\frac{\sqrt{mn}\sin\chi\cos\chi}{\sqrt{1-i^2\sin^2\chi}},$$

or $\Theta=\tau=\tau_{,}$, when $\tau_{,,}=0$, or $\omega=\frac{\pi}{2}$.

LXX. When we take the negative parameter m instead of the positive n, (17.) gives

$$\sigma+\sigma_{,}-\sigma_{,,}=\left(\frac{1-m}{m}\right)\sqrt{mn}\left[\int\frac{d\varphi}{M_{\varphi}\sqrt{I_{\varphi}}}+\int\frac{d\chi}{M_{\chi}\sqrt{I_{\chi}}}-\int\frac{d\omega}{M_{\omega}\sqrt{I_{\omega}}}\right]. \qquad (357.)$$

Now the sum of these arcs is equal to a circular arc$-\Theta_{,}$, which may be determined by the expression

$$\tan\Theta_{,}=\frac{\sqrt{mn}\sin\varphi\sin\chi\sin\omega}{1+\frac{m}{1-m}\cos\varphi\cos\chi\cos\omega}, \qquad (358.)$$

whence $$\sigma+\sigma_{,}-\sigma_{,,}=-\Theta_{,}. \qquad (359.)$$

If we compare together (356.) and (359.), we shall have the following simple rela-

tion between the five circular arcs Θ, $\Theta_{,}$, τ, $\tau_{,}$, $\tau_{,,}$,

$$\Theta+\Theta_{,}=\tau+\tau_{,}-\tau_{,,}. \quad \ldots \ldots \quad (360.)$$

We may give an independent proof of this remarkable theorem.

The primary theorem (338.) $\cos\omega=\cos\phi\cos\chi-\sin\phi\sin\chi\sqrt{I_{\omega}}$

gives
$$\frac{\sin\omega\cos\omega}{\sqrt{I_{\omega}}}=\frac{\sin\phi\sin\chi\sin\omega\cos\omega}{\cos\phi\cos\chi-\cos\omega},$$

and $\cos^2\phi+\cos^2\chi+\cos^2\omega=1+2\cos\phi\cos\chi\cos\omega-i^2\sin^2\phi\sin^2\chi\sin^2\omega$.

Let
$$\sin\phi\sin\chi\sin\omega=\text{U}, \quad \cos\phi\cos\chi\cos\omega=\text{V}. \quad \ldots \ldots \quad (361.)$$

Now
$$\tan\tau_{,,}=\frac{\sqrt{mn}\sin\omega\cos\omega}{\sqrt{1-i^2\sin^2\omega}}=-\frac{\sqrt{mn}\,\text{U}\cos^2\omega}{\cos^2\omega-\text{V}},$$

whence
$$\tan\tau=\frac{\sqrt{mn}\,\text{U}\cos^2\phi}{\cos^2\phi-\text{V}}, \quad \tan\tau_{,}=\frac{\sqrt{mn}\,\text{U}\cos^2\chi}{\cos^2\chi-\text{V}},$$

and
$$\tan(\tau+\tau_{,}-\tau_{,,})=\frac{\tan\tau+\tan\tau_{,}-\tan\tau_{,,}+\tan\tau\tan\tau_{,}\tan\tau_{,,}}{1+\tan\tau_{,,}\tan\tau_{,}+\tan\tau\tan_{,,}-\tan\tau_{,}\tan\tau},$$

whence

$$\tan(\tau+\tau_{,}-\tau_{,,})=\frac{\sqrt{mn}\,\text{U}\left[\frac{\cos^2\phi}{\cos^2\phi-\text{V}}+\frac{\cos^2\chi}{\cos^2\chi-\text{V}}+\frac{\cos^2\omega}{\cos^2\omega-\text{V}}-\frac{mn\text{U}^2\cos^2\phi\cos^2\chi\cos^2\omega}{(\cos^2\phi-\text{V})(\cos^2\chi-\text{V})(\cos^2\omega-\text{V})}\right]}{1-mn\text{U}^2\left[\frac{\cos^2\chi\cos^2\omega}{(\cos^2\chi-\text{V})(\cos^2\omega-\text{V})}+\frac{\cos^2\omega\cos^2\phi}{(\cos^2\omega-\text{V})(\cos^2\phi-\text{V})}+\frac{\cos^2\phi\cos^2\chi}{(\cos^2\phi-\text{V})(\cos^2\chi-\text{V})}\right]}.$$

If we reduce this expression, we shall have, on introducing the relations

$$\left.\begin{aligned}&\cos^2\phi+\cos^2\chi+\cos^2\omega=1+2\text{V}-i^2\text{U}^2,\\ \text{and}\quad&\cos^2\omega\cos^2\chi+\cos^2\phi\cos^2\omega+\cos^2\chi\cos^2\phi=\text{V}^2+2\text{V}+j^2\text{U}^2,\end{aligned}\right\} \quad \ldots \ldots \quad (362.)$$

$$\tan(\tau+\tau_{,}-\tau_{,,})=\frac{[2j^2+(i^2+mn)\text{V}]\sqrt{mn}\,\text{U}}{j^2+(i^2+mn)\text{V}-mn(\text{V}^2+j^2\text{U}^2)}. \quad \ldots \ldots \quad (363.)$$

If we now combine the values of $\tan\Theta$ and $\tan\Theta_{,}$, given in (355.) and (358.), we shall have

$$\tan(\Theta+\Theta_{,})=\frac{[2j^2+(i^2+mn)\text{V}]\sqrt{mn}\,\text{U}}{j^2+(i^2+mn)\text{V}-mn(\text{V}^2+j^2\text{U}^2)}, \quad \ldots \ldots \quad (364.)$$

whence
$$\Theta+\Theta_{,}=\tau+\tau_{,}-\tau_{,,},$$

as is evident from an inspection of the preceding formulæ.

On Conjugate Arcs of a Logarithmic Ellipse.

LXXI. In (162.) substitute χ and ω successively for ϕ. Let

$$\sqrt{z}=\left(\frac{1-n}{n}\right)\sqrt{mn}, \quad \Phi=\frac{\sin\phi\cos\phi\sqrt{I_{\phi}}}{1-n\sin^2\phi}, \quad \text{X}=\frac{\sin\chi\cos\chi\sqrt{I_{\chi}}}{1-n\sin^2\chi}, \quad \Omega=\frac{\sin\omega\cos\omega\sqrt{I_{\omega}}}{1-n\sin^2\omega}, \quad (365.)$$

we shall have, adding the three resulting equations together, and dividing by $\frac{n-m}{\sqrt{mn}}$,

$$\frac{2}{k}[\Sigma_{\omega}-\Sigma_{\chi}-\Sigma_{\phi}]=\frac{\sqrt{mn}}{n-m}\Big[n\Phi+n\text{X}-n\Omega-\Big(\int d\phi\sqrt{I_{\phi}}+\int d\chi\sqrt{I_{\chi}}-\int d\omega\sqrt{I_{\omega}}\Big)\Big]$$
$$-\frac{m}{n}\frac{(1-n)}{(n-m)}\sqrt{mn}\left[\int\frac{d\phi}{\sqrt{I_{\phi}}}+\int\frac{d\chi}{\sqrt{I_{\chi}}}-\int\frac{d\omega}{\sqrt{I_{\omega}}}\right]-\sqrt{z}\left[\int\frac{d\phi}{N_{\phi}\sqrt{I_{\phi}}}+\int\frac{d\chi}{N_{\chi}\sqrt{I_{\chi}}}-\int\frac{d\omega}{N_{\omega}\sqrt{I_{\omega}}}\right]. \quad (366.)$$

Now as φ, χ, and ω are conjugate amplitudes,

$$\int\frac{d\varphi}{\sqrt{I}}+\int\frac{d\chi}{\sqrt{I}}-\int\frac{d\omega}{\sqrt{I}}=0, \text{ and } \int d\varphi\sqrt{I}+\int d\chi\sqrt{I}-\int d\omega\sqrt{I}=i^2\sin\varphi\sin\chi\sin\omega.$$

See HYMER's Integral Calculus, p. 206.

Whence $\frac{2}{k}[\Sigma_\omega-\Sigma_\chi-\Sigma_\varphi]=\frac{\sqrt{mn}}{n-m}[n\Phi+n\mathrm{X}-n\Omega-i^2\sin\varphi\sin\chi\sin\omega]$

$$-\sqrt{\varkappa}\left[\int\frac{d\varphi}{N_\varphi\sqrt{I}}+\int\frac{d\chi}{N_\chi\sqrt{I}}-\int\frac{d\omega}{N_\omega\sqrt{I}}\right]\quad\text{(367.)}$$

We have now to compute the sum of $\Phi+\mathrm{X}-\Omega$.

Since $\sqrt{I_\omega}=\frac{\cos\varphi\cos\chi-\cos\omega}{\sin\varphi\sin\chi}$, $\frac{\sin\omega\cos\omega\sqrt{I_\omega}}{1-n\sin^2\omega}=\Omega=-\frac{\sin^2\omega(\cos^2\omega-V)}{N_\omega U}$, if we make, as before, $\cos\varphi\cos\chi\cos\omega=V$, and $\sin\varphi\sin\chi\sin\omega=U$. Finding similar expressions for Φ and X, we shall have

$$n\Phi+n\mathrm{X}-n\Omega=\frac{n}{U}\left[\frac{\sin^2\varphi\cos^2\varphi}{N_\varphi}+\frac{\sin^2\chi\cos^2\chi}{N_\chi}-\frac{\sin^2\omega\cos^2\omega}{N_\omega}\right]-\frac{V}{U}\left[\frac{n\sin^2\omega}{N_\omega}+\frac{n\sin^2\chi}{N_\chi}+\frac{n\sin^2\varphi}{N_\varphi}\right].\quad\text{(368.)}$$

Now $$\frac{n\sin^2\varphi\cos^2\varphi}{UN}=\frac{\cos^2\varphi(1+n\sin^2\varphi-1)}{NU}=\frac{\cos^2\varphi}{NU}-\frac{\cos^2\varphi}{U},$$

and $$\frac{\cos^2\varphi}{NU}=\frac{1+n-n\sin^2\varphi-1}{nNU}=\frac{1}{nU}-\frac{(1-n)}{nNU},$$

whence $$\frac{n\sin^2\varphi\cos^2\varphi}{NU}=\frac{1}{nU}-\frac{\cos^2\varphi}{U}-\frac{(1-n)}{nNU}, \text{ and } -\frac{Vn\sin^2\varphi}{NU}=\frac{V}{U}-\frac{V}{NU}.$$

Finding similar expressions for the functions of ω and χ, and recollecting that, as in (362.), $\cos^2\varphi+\cos^2\chi+\cos^2\omega=1+2V-i^2U^2$, we shall have, making $W=1-n+nV$,

$$nU(n\Phi+n\mathrm{X}-n\Omega)=3-n+nV+ni^2U^2-W\left[\frac{1}{N_\varphi}+\frac{1}{N_\chi}+\frac{1}{N_\omega}\right],$$

Now $$\int d\varphi\sqrt{I_\varphi}+\int d\chi\sqrt{I_\chi}-\int d\omega\sqrt{I_\omega}=i^2U, \text{ whence}$$

$$nU\left[n\Phi+n\mathrm{X}-n\Omega-\left(\int d\varphi\sqrt{I}+\int d\chi\sqrt{I}-\int d\omega\sqrt{I}\right)\right]=2-W\left[\frac{1}{N_\varphi}+\frac{1}{N_\chi}+\frac{1}{N_\omega}-1\right].\quad\text{(369.)}$$

We shall find, after some complicated calculations, $N_\varphi N_\chi N_\omega=W^2-n^2\varkappa U^2$, . (370.)

and $N_\chi N_\omega+N_\omega N_\varphi+N_\varphi N_\chi=W^2+2W-n(1-n)(i^2+m)U^2$. . . . . (371.)

Substituting the values hence derived, the whole expression becomes divisible by nU^2, and we shall obtain, finally, the following expression,

$$\frac{\sqrt{mn}}{n-m}[n\Phi+n\mathrm{X}-n\Omega-i^2U]=\frac{n\sqrt{\varkappa}WU}{W^2-n^2\varkappa U^2}+\frac{2mn^2\sqrt{\varkappa}UV}{(n-m)(W^2-n^2\varkappa U^2)}.\quad\text{(372.)}$$

It may easily be shown, that

$$-\sqrt{\varkappa}\left[\int\frac{d\varphi}{N_\varphi\sqrt{I}}+\int\frac{d\chi}{N_\chi\sqrt{I}}-\int\frac{d\omega}{N_\omega\sqrt{I}}\right]=\frac{1}{2}\log\left[\frac{1-n+nV+n\sqrt{\varkappa}U}{1-n+nV-n\sqrt{\varkappa}U}\right],\quad\text{(373.)}$$

or writing, as before, W for $1-n+n\mathrm{V}$, and multiplying numerator and denominator by the numerator,

$$-\sqrt{\varkappa}\left[\int\frac{d\varphi}{\mathrm{N}_{\varphi}\sqrt{\mathrm{I}}}+\int\frac{d\chi}{\mathrm{N}_{\chi}\sqrt{\mathrm{I}}}-\int\frac{d\omega}{\mathrm{N}_{\omega}\sqrt{\mathrm{I}}}\right]=\log\left[\frac{\mathrm{W}+n\sqrt{\varkappa}\mathrm{U}}{\sqrt{\mathrm{W}^2-n^2\varkappa\mathrm{U}^2}}\right]. \quad . \quad . \quad . \quad . \qquad (374.)$$

Now let
$$\frac{n\sqrt{\varkappa}\mathrm{U}}{\mathrm{W}}=\sin\xi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (375.)$$

and the preceding logarithm becomes $\log(\sec\xi+\tan\xi)$, which is, we know, the integral of $\int\frac{d\xi}{\cos\xi}$.

Now
$$\frac{n\sqrt{\varkappa}\mathrm{WU}}{\mathrm{W}^2-n^2\varkappa\mathrm{U}^2}=\sec\xi\tan\xi;\ \text{and as } 2\int\frac{d\xi}{\cos^3\xi}=\sec\xi\tan\xi+\int\frac{d\xi}{\cos\xi},$$

we shall have, dividing by 2,

$$\Sigma_{\omega}-\Sigma_{\varphi}-\Sigma_{\chi}=k\int\frac{d\xi}{\cos^3\xi}+\frac{kmn^2\sqrt{\varkappa}\mathrm{UV}}{(n-m)(\mathrm{W}^2-n^2\varkappa\mathrm{U}^2)}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (376.)$$

Hence the sum of three arcs of a logarithmic ellipse may be expressed by an arc of a parabola and a right line.

When one of the arcs Σ_{ω} is a quadrant, $\mathrm{V}=0$, and the equation becomes

$$\left[\Sigma_{\frac{\pi}{2}}-\Sigma_{\chi}\right]-\Sigma_{\varphi}=k\int\frac{d\xi}{\cos^3\xi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (377.)$$

which coincides with (160.).

If we apply to (163.) the same process, step by step, and make $\sin\zeta=\frac{m\sqrt{\varkappa_{\prime}}\mathrm{U}}{\mathrm{W}_{\prime}}$, in which $\mathrm{W}_{\prime}=1-m+m\mathrm{V}$, we shall have

$$\Sigma_{\omega}-\Sigma_{\chi}-\Sigma_{\varphi}=-k\int\frac{d\zeta}{\cos^3\zeta}+\frac{km^2n\sqrt{\varkappa_{\prime}}\mathrm{UV}}{(n-m)(\mathrm{W}_{\prime}^2-m^2\varkappa_{\prime}\mathrm{U}^2)}+k\int\frac{d\tau}{\cos^3\tau}+k\int\frac{d\tau_{\prime}}{\cos^3\tau_{\prime}}-k\int\frac{d\tau_{\prime\prime}}{\cos^3\tau_{\prime\prime}}. \quad . \qquad (378.)$$

If we subtract this equation from (376.), we shall have

$$\int\frac{d\xi}{\cos^3\xi}+\int\frac{d\zeta}{\cos^3\zeta}=\int\frac{d\tau}{\cos^3\tau}+\int\frac{d\tau_{\prime}}{\cos^3\tau_{\prime}}-\int\frac{d\tau_{\prime\prime}}{\cos^3\tau_{\prime\prime}}+\frac{mn}{n-m}\mathrm{UV}\left[\frac{m\sqrt{\varkappa_{\prime}}}{\mathrm{W}_{\prime}^2-m\varkappa_{\prime}\mathrm{U}^2}-\frac{n\sqrt{\varkappa}}{\mathrm{W}^2-n^2\varkappa\mathrm{U}^2}\right]. \quad . \qquad (379.)$$

Now this last term is divisible by $(n-m)$, and may be reduced to the expression

$$\frac{mn\sqrt{mn}\mathrm{UV}[\mathrm{V}^2+j^2mn\mathrm{U}^2-j^2(1-\mathrm{V})^2]}{[\mathrm{W}^2-n^2\varkappa\mathrm{U}^2][\mathrm{W}_{\prime}^2-m^2\varkappa_{\prime}\mathrm{U}^2]}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (380.)$$

If in (170.), which gives the relation between conjugate elliptic integrals of the third order, we substitute successively φ, χ and ω, and add the equations thence resulting, we shall have

$$\int\frac{d\xi}{\cos\xi}+\int\frac{d\zeta}{\cos\zeta}=\int\frac{d\tau}{\cos\tau}+\int\frac{d\tau_{\prime}}{\cos\tau_{\prime}}-\int\frac{d\tau_{\prime\prime}}{\cos\tau_{\prime\prime}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (381.)$$

in which

$$\left.\begin{aligned}&\sin\xi=\frac{\sqrt{mn}\sin\varphi\sin\chi\sin\omega}{1+\frac{n}{1-n}\cos\varphi\cos\chi\cos\omega}, \quad \sin\zeta=\frac{\sqrt{mn}\sin\varphi\sin\chi\sin\omega}{1+\frac{m}{1-m}\cos\varphi\cos\chi\cos\omega}\\&\sin\tau=\frac{\sqrt{mn}\sin\varphi\cos\varphi}{\sqrt{1-i^2\sin^2\varphi}}, \quad \sin\tau_{\prime}=\frac{\sqrt{mn}\sin\chi\cos\chi}{\sqrt{1-i^2\sin^2\chi}}, \quad \sin\tau_{\prime\prime}=\frac{\sqrt{mn}\sin\omega\cos\omega}{\sqrt{1-i^2\sin^2\omega}}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad (382.)$$

If, in these equations, we change n into $-n$, and therefore $\sin\xi$ into $\sqrt{-1}\tan\Theta$, $\sin\zeta$ into $\sqrt{-1}\tan\Theta'$,

$\sin\tau$ into $\sqrt{-1}\tan\tau$, $\sin\tau_{\prime}$ into $\sqrt{-1}\tan\tau_{\prime}$, and $\sin\tau_{\prime\prime}$ into $\sqrt{-1}\tan\tau_{\prime\prime}$,

the preceding equations will become

$$\left.\begin{array}{l} \tan\Theta=\dfrac{\sqrt{mn}\sin\phi\sin\chi\sin\omega}{1-\dfrac{n}{1+n}\cos\phi\cos\chi\cos\omega},\quad \tan\Theta'=\dfrac{\sqrt{mn}\sin\phi\sin\chi\sin\omega}{1+\dfrac{m}{1-m}\cos\phi\cos\chi\cos\omega} \\ \tan\tau=\dfrac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}},\quad \tan\tau_{\prime}=\dfrac{\sqrt{mn}\sin\chi\cos\chi}{\sqrt{1-i^2\sin^2\chi}},\quad \tan\tau_{\prime\prime}=\dfrac{\sqrt{mn}\sin\omega\cos\omega}{\sqrt{1-i^2\sin^2\omega}}. \end{array}\right\} \quad (383.)$$

and $\Theta+\Theta_{\prime}=\tau+\tau_{\prime}-\tau_{\prime\prime}$, as in (360.), values which coincide with those found in LXIX. for the circular form. Or we may pass from the logarithmic to the circular form, or from the paraboloid to the sphere, or inversely, by the imaginary transformations above referred to.

We shall find on trial, that the angles v, v' and τ in (279.) fulfil the condition of conjugate amplitudes.

Section IX.—*On the Maximum Protangent Arcs of Hyperconic Sections.*

LXXII. Since the protangents vanish at the summits of these curves, there must be some intermediate position at which they attain their maximum. When the curve has but one summit, as is the case in the parabola, the hyperbola, the logarithmic parabola, and the logarithmic hyperbola, there evidently can be no maximum*.

In the plane ellipse, the protangent $t=\dfrac{ai^2\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}$. If we differentiate this expression with respect to ϕ, and make the differential coefficient $\dfrac{dt}{d\phi}=0$, we shall get

$$\tan\phi=\frac{1}{\sqrt{j}}. \quad (384.)$$

Substituting this value of $\tan\phi$ in the preceding expression,

$$t=a-b. \quad (385.)$$

In this case, the arcs drawn from the vertices of the curve, and which are compared together, have a common extremity, or they together constitute the quadrant.

The coordinates x, y of the arc measured from the vertex of the minor axis are $x=a\sin\vartheta$, $y=b\cos\vartheta$, therefore $\dfrac{y}{x}=\dfrac{b}{a}\cot\vartheta=j\cot\vartheta$, since $ja=b$. If we now make $\cot\vartheta=\sqrt{j}$, $\dfrac{y}{x}=j^{\frac{3}{2}}$. Again, as $\tan\lambda=\dfrac{a^2}{b^2}\dfrac{y'}{x'}$, $\dfrac{y'}{x'}=j^2\tan\lambda$; or making $\lambda=\vartheta$, or $\tan\lambda=\dfrac{1}{\sqrt{j}}$

* The investigation of these particular values of those portions of the tangent arcs to the curves, which lie between the points of contact and the perpendicular arcs from the origin upon them—or as they have been termed in this paper, protangent arcs—is of importance; because, as we shall show in the next section, in the different series of derived hyperconic sections, the maximum protangent arc of any curve in the series, becomes a parameter in the integral of the curve immediately succeeding.

$\frac{y'}{x'}=j^{\frac{3}{2}}$, or $\frac{y'}{x'}=\frac{y}{x}$. Therefore the arcs have a common extremity. We have also $\tan^2\lambda=\frac{a}{b}$. This property of the plane ellipse, called FAGNANO's theorem, may be found in any elementary treatise on elliptic functions. See HYMER's Integral Calculus, p. 209.

On the Maximum Protangent Arc in the Spherical Hyperconic Section.

LXXIII. If we assume the expression found for this arc τ in (45.),

$$\tan\tau=\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}, \qquad (386.)$$

and differentiate it, as in the last article, and make $\frac{d\tau}{d\phi}=0$, we shall find, as before,

$$\tan\phi=\frac{1}{\sqrt{j}}=\sqrt{\frac{\sin\alpha}{\sin\beta}}. \qquad (387.)$$

If we substitute this value of $\tan\phi$ in the preceding expression, we shall obtain

$$\tan\bar{\tau}=\tan\alpha\sec\beta-\tan\beta\sec\alpha, \qquad (388.)$$

writing $\bar{\tau}$ to denote the maximum protangent.

Now if we turn to Art. LVIII., we shall there find that this equation connects the amplitudes of three conjugate arcs of a plane parabola. Or if $\bar{\tau}$, β, and α are made the three normal angles of a plane parabola, and $(k.\bar{\tau})$, $(k.\beta)$, $(k.\alpha)$ the three corresponding arcs of the parabola, we shall have

$$(k.\alpha)-(k.\beta)-(k.\tau)=k\tan\alpha\tan\beta\tan\tau.$$

If in (386.) we substitute for $\sin\phi$ and $\cos\phi$ their values $\frac{1}{\sqrt{1+j}}$ and $\frac{\sqrt{j}}{\sqrt{1+j}}$, the expression will become

$$\tan\bar{\tau}=\frac{\sqrt{mn}}{(1+j)}. \qquad (389.)$$

We shall see the importance of this value of $\bar{\tau}$ in the next section.

the spherical parabola, as $m=n=i$, $\tan^2\bar{\tau}=\frac{1-j}{1+j}=i_{,}$.

Precisely in the same manner as in the plane ellipse, we may show that when $\tan\tau$ has the preceding value, the arcs drawn from the vertices of the curve have a common extremity. This will be shown by proving that the vector arcs, drawn from the centre of the curve to the extremities of the compared arcs, have the same inclination to the principal arc 2α. Now ψ and ψ' being these inclinations, as in XIV., we find

$$\tan^2\lambda=\frac{\tan^4\alpha}{\tan^4\beta}\tan^2\psi,$$

and (39.) shows that $\tan\phi=\cos\varepsilon\tan\lambda$. Hence reducing,

$$\tan^2\psi=\frac{\tan^2\beta}{\tan^2\alpha}\frac{\sin^2\beta}{\sin^2\alpha}\tan^2\phi. \qquad (a.)$$

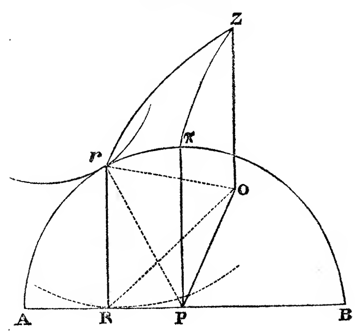
Again, (49.) shows, when we measure the arc from the minor principal arc, that $\cot\theta=\frac{a}{b}\frac{y}{x}$; or $\cot\theta=\frac{\sin\alpha}{\sin\beta}\tan\psi'$. Now in order that we may compare these arcs together, we must have $\theta=\lambda$. Hence

$$\tan^2\psi'=\frac{\tan^2\beta}{\tan^2\alpha}\cdot\frac{1}{\tan^2\varphi}. \quad \text{(b.)}$$

When we substitute for φ any particular value, (a.) and (b.) will give the corresponding values of $\tan\psi$ and $\tan\psi'$; but when we make $\tan^2\varphi=\frac{\sin\alpha}{\sin\beta}=\frac{1}{j}$, the values of ψ and ψ' become equal, or the compared arcs together constitute the quadrant.

LXXIV. To determine the inclination, to the horizontal plane, of the tangent drawn to any point of the spherical ellipse. The spherical ellipse being taken as the curve of intersection of a cylinder by a sphere as in (X.), through a side Rr of the cylinder let a plane be drawn, it will cut the sphere in a *small* circle, which will touch the spherical ellipse in the point r, and will cut the base of the hemisphere in the right line RP, which touches the base of the cylinder at the point R. Let O be the centre of the sphere and Z the centre of the spherical hyperconic. Through the line OZ let a plane be drawn at right angles to the plane of the small circle R$r\pi$P, it will cut the sphere in the arc of a great circle Zπ at right angles to the arc $r\pi$; and as the three planes, namely, the horizontal plane, the plane of the small circle, and the plane of the great circle ZOPπ, are mutually at right angles, the right lines in which they intersect PR, Pπ, PO are mutually at right angles, therefore P is the foot of the perpendicular drawn from the centre O of the base of the cylinder, to the tangent RP which touches the curve. P is also the centre of the small circle A$r\pi$, since AB is a cord of the sphere. Hence Aπ is a quadrant, and therefore, $r\pi$ or ν is the inclination of the element of the spherical ellipse at r to the base of the hemisphere. Now ZO is the radius of the sphere, and Pr that of the small circle. RPO is a right angle, and therefore $\overline{\mathrm{OR}}^2=\overline{\mathrm{OP}}^2+\overline{\mathrm{PR}}^2$. Hence $\overline{\mathrm{R}r}^2=\overline{\mathrm{O}r}^2-\overline{\mathrm{OR}}^2$. Now for the moment putting A and B for the semiaxes of the base of the cylinder, $\overline{\mathrm{OP}}^2=\mathrm{A}^2\cos\lambda+\mathrm{B}^2\sin\lambda$, and

Fig. 26.



$$\overline{\mathrm{RP}}^2=\frac{(\mathrm{A}^2-\mathrm{B}^2)^2\sin^2\lambda_,\cos^2\lambda_,}{\mathrm{A}^2\cos^2\lambda_,+\mathrm{B}^2\sin^2\lambda_,}.\quad \text{Whence } \overline{\mathrm{OR}}^2=\frac{\mathrm{A}^4\cos^2\lambda_,+\mathrm{B}^4\sin^2\lambda_,}{\mathrm{A}^2\cos^2\lambda_,+\mathrm{B}^2\sin^2\lambda_,}, \quad \text{(a.)}$$

and therefore $\overline{\mathrm{R}r}^2=\overline{\mathrm{O}r}^2-\frac{\mathrm{A}^4\cos^2\lambda_,+\mathrm{B}^4\sin^2\lambda_,}{\mathrm{A}^2\cos^2\lambda_,+\mathrm{B}^2\sin^2\lambda_,}$. Let $\mathrm{O}r=1$, $\mathrm{A}=\sin\alpha$, $\mathrm{B}=\sin\beta$, . . (b.)

and as $\tan^2\nu=\frac{\overline{\mathrm{RP}}^2}{\overline{\mathrm{R}r}^2}$, $\tan^2\nu=\frac{(\sin^2\alpha-\sin^2\beta)^2\sin^2\lambda_,\cos^2\lambda_,}{\sin^2\alpha\cos^2\alpha\cos^2\lambda_,+\sin^2\beta\cos^2\beta\sin^2\lambda_,}$.

Let, as in (25.), $\tan\lambda_{\prime}=\cos\varepsilon\tan\phi=\frac{\cos\alpha}{\cos\beta}\tan\phi$. Substituting, we get the expression

$$\tan\nu=\frac{\sin\varepsilon\sin\eta\sin\phi\cos\phi}{\sqrt{(1-\sin^2\varepsilon\sin^2\phi)(1-\sin^2\eta\sin^2\phi)}}. \quad \ldots \quad (390.)$$

In supplemental spherical ellipses, since $\sin\eta$ and $\sin\varepsilon$* are respectively equal to $\sin\varepsilon'$ and $\sin\eta'$, we infer, therefore, that in supplemental spherical ellipses the inclinations to the plane of xy of the tangents to the curves are the same, when the amplitudes ϕ are the same.

If we now differentiate this expression, and make $\frac{d\nu}{d\phi}=0$, we shall find that $\tan^2\phi=\frac{\tan\alpha}{\tan\beta}$. If we substitute this value of $\tan\phi$ in (390.), we shall get

$$\tan\nu=\tan(\alpha-\beta), \text{ or } \nu=\alpha-\beta. \quad \ldots \quad (391.)$$

Hence the maximum inclination to the plane of xy of the tangent to the spherical ellipse is equal to the difference between the principal semiarcs. It is remarkable that the point of the curve which gives the maximum difference between the arcs, which together constitute the quadrant of the spherical ellipse, is not the point of greatest inclination. For this point is found by making $\tan^2\phi=\frac{\tan\alpha}{\tan\beta}$; while the point of maximum difference is obtained by putting $\tan^2\phi=\frac{\sin\alpha}{\sin\beta}$. This is the more worthy of notice, as we shall find the two points—the point of maximum division, and the point of greatest inclination—to coincide in the logarithmic ellipse.

If we take the two plane ellipses which are the projections of the spherical ellipse, one being the perspective, and the other the orthogonal projection, and seek on these plane ellipses their points of maximum division, we shall find that the angles, which the perpendiculars on the tangents, through these points of maximum division of those plane curves, make with the principal arc, are the values which must be assigned to the amplitude ϕ, to determine the point where the tangent to the curve has the greatest inclination to the plane of xy, and the point which divides the quadrant into two parts, such that their difference shall be a maximum. This is plain; for the semiaxes of one ellipse are $k\tan\alpha$, $k\tan\beta$; while the semiaxes of the other are $k\sin\alpha$ and $k\sin\beta$. And these angles are given by the equations

$$\tan^2\lambda=\frac{\tan\alpha}{\tan\beta}; \text{ and } \tan^2\lambda_{\prime}=\frac{\sin\alpha}{\sin\beta}.$$

On the Maximum Protangent in the Logarithmic Ellipse.

LXXV. If we follow the steps previously indicated, and differentiate the expression found in (165),

$$\sin\tau=\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}, \quad \ldots \quad (a.)$$

* Theory of Elliptic Integrals, p. 19.

τ being the normal angle of the tangent parabolic arc to the logarithmic ellipse, this, evidently, will be a maximum when the parabolic arc is a maximum. Put the differential coefficient $\frac{d\tau}{d\varphi}=0$. This gives, as before, $\tan\varphi=\frac{1}{\sqrt{j}}$. Substituting this expression in (a.), we get

$$\sin\bar{\tau}=\frac{\sqrt{mn}}{(1+j)}. \quad \ldots \ldots \ldots \quad (392.)$$

We shall find the importance of this expression in the next section.

From (392.) we derive $\qquad \tan^2\bar{\tau}=\frac{mn}{(1+j)^2-mn}$.

Now $\qquad (1+j)^2=2+2j-i^2=2+2j-m-n+mn.$ Hence as

$$j=\sqrt{(1-m)(1-n)},\ (1+j)^2-mn=[\sqrt{1-m}+\sqrt{1-n}]^2.$$

Whence we get $\tan\bar{\tau}=\frac{\sqrt{mn}}{\sqrt{1-m}+\sqrt{1-n}}$. Multiply this equation, numerator and denominator by $\sqrt{1-m}-\sqrt{1-n}$, and the last expression will become

$$\tan\bar{\tau}=\frac{\sqrt{mn}\sqrt{1-m}}{n-m}-\frac{\sqrt{mn}\sqrt{1-n}}{n-m}.$$

In (171.) we found for the semiaxes of the cylinder, whose intersection with the paraboloid is the logarithmic ellipse, $\frac{a}{k}=\frac{\sqrt{mn}\sqrt{1-m}}{n-m}$, $\quad \frac{b}{k}=\frac{\sqrt{mn}\sqrt{1-n}}{n-m}$.

Hence $$\tan\bar{\tau}=\left(\frac{a-b}{k}\right). \quad \ldots \ldots \ldots \quad (393.)$$

This gives a simple expression for the tangent of the maximum parabolic arc, analogous to (385.) and (391.). We have only to take in the parabola, whose semiparameter is k, an arc whose ordinate is $a-b$, to determine the maximum protangent parabolic arc.

The value $\tan\varphi=\frac{1}{\sqrt{j}}$, which fixes the position and magnitude of the maximum protangent arc to the logarithmic ellipse, renders $\tan^2\lambda=\frac{a}{b}$. For (150.) gives $\tan^2\varphi=\frac{\alpha+\beta}{\alpha}\tan^2\lambda$. But (152.) gives $\frac{\alpha+\beta}{\alpha}=\frac{C}{C-B}$, and $\frac{C}{C-B}=\frac{1}{1-m}$, hence $\tan^2\varphi=\frac{\tan^2\lambda}{1-m}$. If we now make

$$\tan^2\varphi=\frac{1}{j}=\frac{1}{\sqrt{(1-n)(1-m)}} \qquad \tan^2\lambda=\sqrt{\frac{1-m}{1-n}}=\frac{a}{b},$$

as we may infer from (171.). Now substituting this value of $\tan^2\lambda$ in (155.), we shall get

$$\tan\tau=\frac{a-b}{k}.$$

Again, if we differentiate the values of x, y, z given in (158.), the coordinates of the extremity of the arc measured from the minor axis, and substitute them in the general

expression for the tangent of the inclination of any curve to the plane of xy, namely, $\frac{dz}{\sqrt{dx^2+dy^2}}$, and make $\vartheta=\lambda$, we shall get, as before, putting for $\tan^2\lambda=\tan^2\vartheta$, the value $\frac{a}{b}$, $\frac{dz}{\sqrt{dx^2+dy^2}}=\frac{a-b}{k}$. Hence the arcs have a common extremity, since they have the same inclination to the plane of xy. As $\frac{a}{b}=\tan^2\lambda$ is the value of $\tan^2\lambda$, which gives the maximum protangent $=a-b$ in the plane ellipse, the base of the cylinder; it follows that the point of maximum division on the logarithmic ellipse is orthogonally projected into the point of maximum division on the plane ellipse; and the corresponding protangent in the latter $a-b$ is the ordinate of the parabolic arc, which expresses the difference between the corresponding arcs of the former. Thus, while the arcs which together constitute the quadrant on the plane ellipse, differ by the difference of the semiaxes $a-b$, the corresponding arcs of the logarithmic ellipse will differ by an arc of a parabola whose ordinate is $a-b$.

LXXVI. When the amplitude φ is given by the equation $\tan\varphi=\frac{1}{\sqrt{j}}$, or when the protangent is a maximum, the corresponding arc of the spherical ellipse, or of the logarithmic ellipse, may be expressed by functions of the first and second orders only. This may be shown as follows. When $\tan\varphi=\frac{1}{\sqrt{j}}$ the arcs σ and $\sigma_{\prime}$ of the spherical ellipse, or the arcs Σ and S of the logarithmic ellipse, together make up the quadrant C. Hence $\sigma+\sigma_{\prime}=C$, or $\Sigma+S=C$. But we have also $\sigma_{\prime}-\sigma=\tau$, as in (52.), and $S-\Sigma=\tau$, as in (160.). Therefore

$$\sigma=\frac{C-\tau}{2} \quad \sigma_{\prime}=\frac{C+\tau}{2}, \quad S=\frac{C+\tau}{2}, \quad \Sigma=\frac{C-\tau}{2}.$$

Or σ and $\sigma_{\prime}$, or Σ and S may be expressed as simple functions of C and τ. Now C, the quadrant, as we have shown in the last section, may be expressed by functions of the first and second orders only, while τ is an arc either of a circle or of a parabola. Hence an elliptic integral of the third order, whose amplitude $\varphi=\tan^{-1}\left(\frac{1}{\sqrt{j}}\right)$, may be expressed by functions of the first and second orders only.

SECTION X.—*On Derivative Hyperconic Sections.*

LXXVII. We shall now proceed to show that, when a hyperconic section is given, whether it be spherical or paraboloidal, we may from it derive a series of curves, whose moduli and parameters shall decrease or increase according to a certain law; so that ultimately the rectification of these curves may be reduced to the calculation of circular or parabolic arcs, or in other words, to circular functions or logarithms. We shall also show that all these derived curves, together with the original curve, may be traced on the *same* generating surface, *i. e.* on the same sphere or paraboloid.

In (186.) we have shown that the rectification of a plane ellipse whose semiaxes are a and b, may be reduced to the rectification of another plane ellipse whose semiaxes $a_{,}$, $b_{,}$ are given by the equations $a_{,}=a+b$, $b_{,}=2\sqrt{ab}$, of which the eccentricity is less than that of the former. $a+b$ is that portion of the tangent, drawn through the point of maximum division, which lies between the axes; and $\sqrt{ab}$ is the perpendicular from the centre on it.

We have shown in (63.) and (74.), that if φ and ψ are connected by the equation

$$\tan(\psi-\varphi)=j\tan\varphi;\text{ while } i \text{ and } i_{,} \text{ are so related, that } i_{,}=\frac{1-\sqrt{1-i^2}}{1+\sqrt{1-i^2}}=\frac{1-j}{1+j},$$

we shall have

$$\int\frac{d\varphi}{\sqrt{1-i^2\sin^2\varphi}}=\frac{(1+i_{,})}{2}\int\frac{d\psi}{\sqrt{1-i_{,}^2\sin^2\psi}}=\frac{(1+i_{,})}{2}\int\frac{d\psi}{\sqrt{\mathrm{I}_{,}}}.$$

Let us now introduce this suggested transformation into the elliptic integral of the third order, *circular* form and *negative* parameter. In (191.) we found

$$2\sin^2\varphi=1+i_{,}\sin^2\psi-\cos\psi\sqrt{\mathrm{I}_{,}}.$$

Now
$$\int\frac{d\varphi}{\mathrm{M}\sqrt{\mathrm{I}}}=\int\frac{d\varphi}{[1-m\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}.$$

Or replacing φ by its equivalent functions in ψ, and recollecting that $m-n+mn=i^2$, since m and n are conjugate parameters, we shall find

$$\int\frac{d\varphi}{\mathrm{M}\sqrt{\mathrm{I}}}=(1+i_{,})\int\frac{d\psi}{[2-m-mi_{,}\sin^2\psi+m\cos\psi\sqrt{\mathrm{I}_{,}}]\sqrt{\mathrm{I}_{,}}}. \quad \ldots \quad (394.)$$

We may eliminate the radical $m\cos\psi\sqrt{\mathrm{I}_{,}}$ from the denominator of this expression, by treating it as the sum of two terms.

Multiplying and dividing the function by their difference, since $1+i_{,}=\frac{2}{1+j}$,

$$4(1-m)\int\frac{d\varphi}{\mathrm{M}\sqrt{\mathrm{I}}}=(1+i_{,})\int\frac{d\psi[2-m-mi_{,}\sin^2\psi-m\cos\psi\sqrt{\mathrm{I}_{,}}]}{\left[1+\frac{mn}{(1+j)^2}\sin^2\psi\right]\sqrt{\mathrm{I}_{,}}}. \quad \ldots \quad (395.)$$

Now it is truly remarkable that whether the parameter of the original function we start from be positive or negative, the parameter of the first derived integral will always be positive. Indeed it is necessary that this should be the case, because the parameters of the derived functions, increasing or diminishing as they do, must at length pass from between the limits 1 and i^2. Should they do so, the integral would be no longer of the circular form, but of the logarithmic. Now we cannot pass from one of these forms to the other by any but an imaginary transformation. This objection does not hold when the parameter is positive, because the limits of the positive parameter are 0 and ∞. It is, too, worthy of remark, that the first derived parameter is always the same, whether we transform from positive or negative parameters. Write

$$n_{,}=\frac{mn}{(1+j)^2}, \quad \ldots \quad (396.)$$

$n_{,}$ is the first derived parameter.

We may transform (395.) into

$$4(1-m)\int\frac{d\phi}{M\sqrt{I}}=(1+i_{,})\int\frac{d\psi\left[2-m-\frac{mi_{,}}{n_{,}}(1+n_{,}\sin^2\psi-1)-m\cos\psi\sqrt{I}\right]}{[1+n_{,}\sin^2\psi]\sqrt{I_{,}}}.$$

Now $\qquad \frac{mi_{,}}{n_{,}}=\frac{i^2}{n},$ and $2-m+\frac{mi_{,}}{n_{,}}=\frac{m+n}{n}$. . . . . (397.)

Hence

$$2\frac{(1-m)}{m}\int\frac{d\phi}{M\sqrt{I}}=\frac{(m+n)\sqrt{n_{,}}}{mn.\sqrt{mn}}\int\frac{d\psi}{[1+n_{,}\sin^2\psi]\sqrt{I_{,}}}-\frac{(1+i_{,})}{2}\frac{i^2}{mn}\int\frac{d\psi}{\sqrt{I}}-\frac{(1+i_{,})}{2\sqrt{n_{,}}}\tan^{-1}(\sqrt{n_{,}}\sin\psi). \quad (398.)$$

We shall now show that $\qquad \frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}=\sqrt{n_{,}}\sin\psi$. . . . . . . (399.)

If we revert to (189.) and (193.), we there find

$$2\sin\phi\cos\phi=\sin\psi[\sqrt{I_{,}}+i_{,}\cos\psi], \text{ and } 2\sqrt{I}=(1+j)[\sqrt{I_{,}}+i_{,}\cos\psi].$$

(396.) gives $\sqrt{mn}=\sqrt{n_{,}}(1+j)$; therefore $\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{I}}=\sqrt{n_{,}}\sin\psi$.

If we replace $\frac{(1+i_{,})}{2}\int\frac{d\psi}{\sqrt{I_{,}}}$ in the preceding equation by its value $\int\frac{d\phi}{\sqrt{I}}$, and put $N_{,}$ for $1+n_{,}\sin^2\psi$,

$$2\left(\frac{1-m}{m}\right)\int\frac{d\phi}{M\sqrt{I}}=\frac{(m+n)}{mn}\sqrt{\frac{n_{,}}{mn}}\int\frac{d\psi}{N_{,}\sqrt{I_{,}}}-\frac{i^2}{mn}\int\frac{d\phi}{\sqrt{I}}-\frac{1}{\sqrt{mn}}\tan^{-1}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{I}}\right]. \quad (400.)$$

Now the common formula for comparing circular integrals with conjugate parameters is, we know, see (47.),

$$\left(\frac{1+n}{n}\right)\int\frac{d\phi}{N\sqrt{I}}-\left(\frac{1-m}{m}\right)\int\frac{d\phi}{M\sqrt{I}}=\frac{i^2}{mn}\int\frac{d\phi}{\sqrt{I}}+\frac{1}{\sqrt{mn}}\tan^{-1}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}\right].$$

Adding these equations we obtain this new formula

$$\left(\frac{1+n}{n}\right)\sqrt{mn}\int\frac{d\phi}{N\sqrt{I}}+\left(\frac{1-m}{m}\right)\sqrt{mn}\int\frac{d\phi}{M\sqrt{I}}=\left(\frac{m+n}{mn}\right)\sqrt{n_{,}}\int\frac{d\psi}{N_{,}\sqrt{I_{,}}}. \quad (401.)$$

By the help of this important formula we may establish a simple relation between the sum of the original conjugate functions of the third order, and the first derived function of this order.

LXXVIII. If σ be the arc of a spherical ellipse, it is shown in (46.) that

$$\sigma=\left(\frac{1+n}{n}\right)\sqrt{mn}\int\frac{d\phi}{N\sqrt{I}}-\frac{i^2}{\sqrt{mn}}\int\frac{d\phi}{\sqrt{I}}-\tan^{-1}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}\right],$$

and in (17.) that $\qquad \sigma=\left(\frac{1-m}{m}\right)\sqrt{mn}\int\frac{d\phi}{M\sqrt{I}}.$

Adding these equations together, and introducing the relation just now established,

$$2\sigma=\frac{(m+n)}{mn}\sqrt{n_{,}}\int\frac{d\psi}{N_{,}\sqrt{I_{,}}}-\frac{i^2}{\sqrt{mn}}\int\frac{d\phi}{\sqrt{I}}-\tan^{-1}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}\right]. \quad (402.)$$

Now as $m-n=i^2-mn$, $\quad (m+n)^2=i^4-2i^2mn+m^2n^2+4mn$.

We have also $mn=n_{\prime}(1+j)^2$, $i_{\prime}=\frac{1-j}{1+j}$, and $i^2=(1-j)(1+j)$, hence

$$m+n=(1+j)^2(1+n_{\prime})\sqrt{m_{\prime}}, \quad (403.)$$

and therefore
$$\left(\frac{m+n}{mn}\right)\sqrt{n_{\prime}}=\left(\frac{1+n_{\prime}}{n_{\prime}}\right)\sqrt{m_{\prime}n_{\prime}}. \quad (404.)$$

It is worthy of especial remark that this coefficient of $\int\frac{d\psi}{N_{\prime}\sqrt{I_{\prime}}}$ is precisely the same in form as the coefficient of $\int\frac{d\phi}{N\sqrt{I}}$.

The preceding equation (402.) may now be written,

$$2\sigma=\left(\frac{1+n_{\prime}}{n_{\prime}}\right)\sqrt{m_{\prime}n_{\prime}}\int\frac{d\psi}{N_{\prime}\sqrt{I_{\prime}}}-\frac{i^2}{\sqrt{mn}}\int\frac{d\phi}{\sqrt{I}}-\tan^{-1}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{I}}\right]. \quad (405.)$$

Let $\sigma_{\prime}$, $n_{\prime}$, $i_{\prime}$, ψ be analogous quantities for the derived spherical ellipse $\sigma_{\prime}$,

$$\sigma_{\prime}=\left(\frac{1+n_{\prime}}{n_{\prime}}\right)\sqrt{m_{\prime}n_{\prime}}\int\frac{d\psi}{N_{\prime}\sqrt{I_{\prime}}}-\frac{i^2_{\prime}}{\sqrt{m_{\prime}n_{\prime}}}\int\frac{d\psi}{\sqrt{I_{\prime}}}-\tan^{-1}\left[\frac{\sqrt{m_{\prime}n_{\prime}}\sin\psi\cos\psi}{\sqrt{I_{\prime}}}\right]. \quad (406.)$$

Let q, $q_{\prime}$, $q_{\prime\prime}$, $q_{\prime\prime\prime}$, &c. denote $\frac{i^2}{\sqrt{mn}}$, $\frac{i^2_{\prime}}{\sqrt{m_{\prime}n_{\prime}}}$, $\frac{i^2_{\prime\prime}}{\sqrt{m_{\prime\prime}n_{\prime\prime}}}$, &c., and put r, $r_{\prime}$, $r_{\prime\prime}$, $r_{\prime\prime\prime}$, &c. for $(1+j)$, $(1+j)(1+j_{\prime})$, $(1+j)(1+j_{\prime})(1+j_{\prime\prime})$, $(1+j)(1+j_{\prime})(1+j_{\prime\prime})(1+j_{\prime\prime\prime})$, &c. Let also Ω, Ψ, $\Psi_{\prime}$, $\Psi_{\prime\prime}$, &c. denote the arcs, whose tangents are

$$\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}},\quad \frac{\sqrt{m_{\prime}n_{\prime}}\sin\psi\cos\psi}{\sqrt{1-i^2_{\prime}\sin^2\psi}},\quad \frac{\sqrt{m_{\prime\prime}n_{\prime\prime}}\sin\psi_{\prime}\cos\psi_{\prime}}{\sqrt{1-i^2_{\prime\prime}\sin^2\psi_{\prime}}},\ \&c.$$

Making these substitutions, and writing Q, $Q_{\prime}$, $Q_{\prime\prime}$, &c. for the coefficients of

$\int\frac{d\phi}{N\sqrt{I}}$, $\int\frac{d\psi}{N_{\prime}\sqrt{I_{\prime}}}$, $\int\frac{d\psi_{\prime}}{N_{\prime\prime}\sqrt{I_{\prime\prime}}}$, (405.) and (406.) become

$$2\sigma=Q_{\prime}\int\frac{d\psi}{N_{\prime}\sqrt{I_{\prime}}}-q\int\frac{d\phi}{\sqrt{I}}-\Omega \quad (a.) \qquad \sigma_{\prime}=Q_{\prime}\int\frac{d\psi}{N_{\prime}\sqrt{I_{\prime}}}-q_{\prime}r\int\frac{d\phi}{\sqrt{I}}-\Psi. \quad (a_{\prime}.)$$

Taking the derivatives of these expressions, we may write

$$2\sigma_{\prime}=Q_{\prime\prime}\int\frac{d\psi_{\prime}}{N_{\prime\prime}\sqrt{I_{\prime\prime}}}-q_{\prime}r\int\frac{d\phi}{\sqrt{I}}-\Psi. \quad (b.) \qquad \sigma_{\prime\prime}=Q_{\prime\prime}\int\frac{d\psi_{\prime}}{N_{\prime\prime}\sqrt{I_{\prime\prime}}}-q_{\prime\prime}r_{\prime}\int\frac{d\phi}{\sqrt{I}}-\Psi_{\prime}. \quad (b_{\prime}.)$$

$$2\sigma_{\prime\prime}=Q_{\prime\prime\prime}\int\frac{d\psi_{\prime\prime}}{N_{\prime\prime\prime}\sqrt{I_{\prime\prime\prime}}}-q_{\prime\prime}r_{\prime}\int\frac{d\phi}{\sqrt{I}}-\Psi_{\prime}. \quad (c.) \qquad \sigma_{\prime\prime\prime}=Q_{\prime\prime\prime}\int\frac{d\psi_{\prime\prime}}{N_{\prime\prime\prime}\sqrt{I_{\prime\prime\prime}}}-q_{\prime\prime\prime}r_{\prime\prime}\int\frac{d\phi}{\sqrt{I}}-\Psi_{\prime\prime}. \quad (c_{\prime}.)$$

Subtract $(a_{\prime}.)$ from (a.), $(b_{\prime}.)$ from (b.), and $(c_{\prime}.)$ from (c.), the integrals of the third order disappear, and we shall have

$$\left.\begin{aligned}
2\sigma-\sigma_{\prime}&=(q_{\prime}r-q)\int\frac{d\phi}{\sqrt{I}}+\Psi-\Omega\\
2\sigma_{\prime}-\sigma_{\prime\prime}&=(q_{\prime\prime}r_{\prime}-q_{\prime}r)\int\frac{d\phi}{\sqrt{I}}+\Psi_{\prime}-\Psi\\
2\sigma_{\prime\prime}-\sigma_{\prime\prime\prime}&=(q_{\prime\prime\prime}r_{\prime\prime}-q_{\prime\prime}r_{\prime})\int\frac{d\phi}{\sqrt{I}}+\Psi_{\prime\prime}-\Psi_{\prime}\\
2\sigma_{\prime\prime\prime}-\sigma_{\prime\prime\prime\prime}&=(q_{\prime\prime\prime\prime}r_{\prime\prime\prime}-q_{\prime\prime\prime}r_{\prime\prime})\int\frac{d\phi}{\sqrt{I}}+\Psi_{\prime\prime\prime}-\Psi_{\prime\prime}
\end{aligned}\right\} \quad (407.)$$

If we add these equations together,

$$\sigma+\sigma_{\prime}+\sigma_{\prime\prime}+\sigma_{\prime\prime\prime}+(\sigma-\sigma_{\prime\prime\prime\prime})=(q_{\prime\prime\prime\prime}r_{\prime\prime\prime}-q)\int\frac{d\varphi}{\sqrt{I}}+\Psi_{\prime\prime\prime}-\Omega. \quad . \quad . \quad . \quad . \quad . \quad . \quad (408.)$$

If we multiply the first of (407.) by 2^3, the second by 2^2, the third by 2, and the fourth by 2^0, and add the results,

$$2^4\sigma-\sigma_{\prime\prime\prime\prime}=(q_{\prime\prime\prime\prime}r_{\prime\prime\prime}+q_{\prime\prime\prime}r_{\prime\prime}+2q_{\prime\prime}r_{\prime}+4q_{\prime}r-8q)\int\frac{d\varphi}{\sqrt{I}}+(\Psi_{\prime\prime\prime}+\Psi_{\prime\prime}+2\Psi_{\prime}+4\Psi-8\Omega), \quad . \quad (409.)$$

an integral which enables us to approximate with ease to the value of the integral of the third order and circular form, in terms of an integral of the first order.

We have shown in XXVIII. how the integral of the first order may be reduced.

The above expressions may be reduced to simpler forms, when the functions are complete. In this case $\Omega=0$, $\Psi=0$, $\Psi_{\prime}=0$, $\Psi_{\prime\prime}=0$, &c.; and when σ is a quadrant, $\sigma_{\prime}$ will be two quadrants, $\sigma_{\prime\prime}$ will be four quadrants, $\sigma_{\prime\prime\prime}$ will be eight quadrants, and so on; the preceding expression may now be written, denoting a quadrant by the symbol $\breve{\sigma}$,

$$16(\breve{\sigma}-\breve{\sigma}_{\prime\prime\prime\prime})=(q_{\prime\prime\prime\prime}r_{\prime\prime\prime}+q_{\prime\prime\prime}r_{\prime\prime}+2q_{\prime\prime}r_{\prime}+4q_{\prime}r-8q)\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{I}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (410.)$$

In (396.) we found for the parameter of the derived integral of the third order, the expression $n_{\prime}=\frac{mn}{(1+j)^2}$. Or, referring to the geometrical representatives of these expressions, we found for the focal distance $\varepsilon_{\prime}$ of this derived curve, the expression $n_{\prime}=\tan^2\varepsilon_{\prime}=\frac{mn}{(1+j)^2}$; but if we turn to (389.) we shall see that this is the expression for the maximum protangent to the original spherical ellipse, which is given by the equation $\tan^2\tau=\frac{mn}{(1+j)^2}$. We thus arrive at this curious relation between the curves successively derived, that the maximum protangent of any one of the spherical ellipses becomes the focal distance of the one immediately succeeding in the series.

LXXIX. Given m, n and i, we may determine $m_{\prime}$, $n_{\prime}$ and $i_{\prime}$,

for $i_{\prime}=\frac{1-j}{1+j}$, $n_{\prime}=\frac{mn}{(1+j)^2}$. Substituting these values of $i_{\prime}$ and $n_{\prime}$ in the equation which connects the parameters, $m_{\prime}-n_{\prime}+m_{\prime}n_{\prime}=i_{\prime}^2$,

$$m_{\prime}=\left[\frac{\sqrt{1+n}-\sqrt{1-m}}{\sqrt{1+n}+\sqrt{1-m}}\right]^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (411.)$$

Hence given m, n and i, we can easily compute the values of $m_{\prime}$, $n_{\prime}$ and $i_{\prime}$, and then of $m_{\prime\prime}$, $n_{\prime\prime}$ and $i_{\prime\prime}$; and so on as far as we please.

Given the semiaxes a and b of the elliptic cylinder, whose intersection with the sphere is the original spherical ellipse, to determine the semiaxes $a_{\prime}$ and $b_{\prime}$ of the cylinder, whose intersection with the sphere shall be the first derived spherical ellipse.

We may derive from (53.) and (54.) the values of a and b in terms of m, n and i, or eliminating i, in terms of m and n only. Now

$$\frac{a^2}{k^2}=\frac{n}{m(1+n)},\ \frac{b^2}{k^2}=\frac{n(1-m)}{m}. \quad \text{Hence } \frac{a_{\prime}^2}{k^2}=\frac{n_{\prime}}{m_{\prime}(1+n_{\prime})},\ \frac{b_{\prime}^2}{k^2}=\frac{n_{\prime}(1-m_{\prime})}{m_{\prime}}.$$

Or substituting the values of $m_{,}$ and $n_{,}$ in terms of m and n, and therefore of a and b,

$$a_{,}=\frac{a+b}{1+\frac{ab}{k^2}},\ b_{,}=\frac{2\sqrt{ab}}{1+\frac{ab}{k^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (412.)$$

When the radius of the sphere is infinite, or the derived curve is a plane ellipse, $a_{,}=a+b$, $b_{,}=2\sqrt{ab}$, as in LXXVII.

When $m=n=i$; $m_{,}=n_{,}=i_{,}$, or when the given curve is a spherical parabola, the derived curve will also be a spherical parabola. Hence all the curves of the series will be spherical parabolas.

If we take the corresponding integral of the third order with a reciprocal parameter l, such that $lm=i^2$, and derive by the foregoing process the first derived function of the third order, we shall find the parameter $l_{,}$ of this function to be positive also, and reciprocal to $n_{,}$, so that $l_{,}\,n_{,}=i_{,}^2$.

Hence, if we deduce a series of derived functions from two primitive functions of the third order and circular form, having either *positive* or *negative* reciprocal parameters, the parameters of all the derived functions $l_{,}$, $l_{,,}$, $l_{,,,}$, $n_{,}$, $n_{,,}$, $n_{,,,}$, will be *positive*, and reciprocal in pairs, so that $l_{,}n_{,}=i_{,}^2$, $l_{,,}n_{,,}=i_{,,}^2$, $l_{,,,}n_{,,,}=i_{,,,}^2$, &c.

LXXX. We may apply the same method of proceeding to the logarithmic ellipse, or to the logarithmic integral of the third order,

$$\int\frac{d\phi}{(1-m\sin^2\phi)\sqrt{1-i^2\sin^2\phi}},\ \text{in which } i^2>m.$$

If on this function we perform the operations effected on the similar integral in (394.), we shall have, after like reductions,

$$\int\frac{d\phi}{M\sqrt{I}}=\frac{(1+i_{,})}{4(1-m)}\int\frac{d\psi[2-m-mi_{,}\sin^2\psi-m\cos\psi\sqrt{I_{,}}]}{[1-m_{,}\sin^2\psi]\sqrt{I_{,}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (413.)$$

We must recollect that

$$M=1-m\sin^2\phi,\ M_{,}=1-m_{,}\sin^2\psi,\ I=1-i^2\sin^2\phi,\ I_{,}=1-i_{,}^2\sin^2\psi,\ \text{and } m_{,}=\frac{mn}{(1+j)^2}. \quad (414.)$$

We may reduce this expression.

The numerator may be put under the form

$$2-m+\frac{mi_{,}}{m_{,}}\{1-m_{,}\sin^2\psi-1\}-m\cos\psi\sqrt{I_{,}}.$$

Now $\qquad 2-m-\frac{mi_{,}}{m_{,}}=\frac{(n-m)}{n}$, and $\frac{mi_{,}}{m_{,}}=\frac{i^2}{n}$. We have also $\frac{\sqrt{i_{,}}}{i}=\frac{1}{1+j}$.

Hence, making the necessary transformations,

$$2\frac{(1-m)}{m}\int\frac{d\phi}{M\sqrt{I}}=\frac{(n-m)}{mn}\frac{\sqrt{i_{,}}}{i}\int\frac{d\psi}{M_{,}\sqrt{I_{,}}}+\frac{i\sqrt{i_{,}}}{mn}\int\frac{d\psi}{\sqrt{_{,}I}}-\frac{\sqrt{i_{,}}}{i}\int\frac{\cos\psi d\psi}{M_{,}}.$$

If into this expression we introduce the relation given in (74.), $\int\frac{d\phi}{\sqrt{I}}=\frac{(1+i_{,})}{2}\int\frac{d\psi}{\sqrt{I_{,}}}$,

we shall have $2\left(\frac{1-m}{m}\right)\int\frac{d\phi}{M\sqrt{I}}=\frac{(n-m)}{mn}\frac{\sqrt{i_{,}}}{i}\int\frac{d\psi}{M_{,}\sqrt{I_{,}}}+\frac{i^2}{mn}\int\frac{d\phi}{\sqrt{I}}-\frac{\sqrt{i}}{i}\int\frac{\cos\psi d\psi}{M_{,}}. \quad . \quad . \quad (415.)$

Now in (399.) it has been shown that $\sqrt{m_{,}}\sin\psi=\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{I}}$, and as $\sqrt{mn}=\sqrt{m_{,}}(1+j)$, the last term of the preceding equation may be written

$$\frac{1}{\sqrt{mn}}\int\frac{\frac{d}{d\phi}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{I}}\right]d\phi}{1-\frac{mn\sin^2\phi\cos^2\phi}{I}}.$$

Substituting this value in the preceding equation and comparing it with (169.) or (170.), we shall find

$$\left(\frac{1-m}{m}\right)\int\frac{d\phi}{M\sqrt{I}}-\left(\frac{1-n}{n}\right)\int\frac{d\phi}{N\sqrt{I}}=\frac{(n-m)}{nm}\frac{\sqrt{i_{,}}}{i}\int\frac{d\psi}{M_{,}\sqrt{I_{,}}}. \quad . \quad . \quad . \quad . \quad . \quad (416.)$$

This equation is analogous to (401.). By the help of it and the last equation we can always express

$$\int\frac{d\phi}{M\sqrt{I}} \text{ or } \int\frac{d\phi}{N\sqrt{I}} \text{ in terms of } \int\frac{d\psi}{M_{,}\sqrt{I_{,}}}.$$

Since $m_{,}=\frac{mn}{(1+j)^2}$ is symmetrical with respect to n and m, we should have obtained the same value for the derived parameter had it been deduced from $\int\frac{d\phi}{N\sqrt{I}}$ instead of $\int\frac{d\phi}{M\sqrt{I}}$.

Since $$m_{,}=\frac{mn}{(1+j)^2},\quad n_{,}=\frac{(1-j)^2-mn}{(1+j)^2-mn},\quad \text{or } n_{,}=\left[\frac{\sqrt{1-m}-\sqrt{1-n}}{\sqrt{1-m}+\sqrt{1-n}}\right]^2. \quad . \quad . \quad . \quad (417.)$$

LXXXI. We may express $m_{,}$ and $n_{,}$ simply, in terms of a and b, the semiaxes of the base of the elliptic cylinder, whose curve of section with the paraboloid is the logarithmic ellipse.

In (171.) we have found the values of m and n in terms of a, b and k, namely,

$$\frac{a}{k}=\frac{\sqrt{mn(1-m)}}{n-m},\quad \frac{b}{k}=\frac{\sqrt{mn(1-n)}}{n-m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a.)$$

Hence $\frac{a-b}{a+b}=\frac{\sqrt{1-m}-\sqrt{1-n}}{\sqrt{1-m}+\sqrt{1-n}}$, or assuming the value of $n_{,}$ in (417.) $n_{,}=\left(\frac{a-b}{a+b}\right)^2$.

Now $$n-m=(1-m)-(1-n)=(\sqrt{1-m}+\sqrt{1-n})(\sqrt{1-m}-\sqrt{1-n}).$$

Or as $$m_{,}=\frac{mn}{(1+j)^2},\quad 1-m_{,}=\frac{(1+j)^2-mn}{(1+j)^2}=\frac{(\sqrt{1-m}+\sqrt{1-n})^2}{(1+j)^2},$$

and (a.) gives $$\frac{a-b}{k}=\frac{\sqrt{mn}}{\sqrt{1-m}+\sqrt{1-n}},$$

therefore $$\frac{1-m_{,}}{m_{,}}=\frac{(\sqrt{1-m}-\sqrt{1-n})^2}{mn}.$$ Hence reducing, $m_{,}=\frac{(a-b)^2}{k^2+(a-b)^2}$.

If we now compare together these expressions for $m_{,}$ and $n_{,}$, namely,

$$m_{,}=\frac{(a-b)^2}{k^2+(a-b)^2},\quad n_{,}=\left(\frac{a-b}{a+b}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (418.)$$

we shall find that $n_{,}>m_{,}$, so long as $k>2\sqrt{ab}$; that when $k=2\sqrt{ab}$, $m_{,}=n_{,}$; and that when $k<2\sqrt{ab}$, $n_{,}<m_{,}$.

To determine the axes of the base of the cylinder, whose intersection with the paraboloid gives the derived logarithmic ellipse.

Since $\frac{a_{\prime}^2}{k^2}=\frac{m_{\prime}n_{\prime}(1-m_{\prime})}{(n_{\prime}-m_{\prime})^2}$, $\frac{b_{\prime}^2}{k^2}=\frac{m_{\prime}n_{\prime}(1-n_{\prime})}{(n_{\prime}-m_{\prime})^2}$, as we may infer from (171.),

we shall have, substituting the preceding values of $m_{\prime}$ and $n_{\prime}$,

$$\frac{a_{\prime}^2}{k^2}=\frac{(a+b)^2k^2}{[k^2-4ab]^2},\quad \frac{b_{\prime}^2}{k^2}=\frac{4ab[k^2+(a-b)^2]}{[k^2-4ab]^2},\quad \text{and } i_{\prime}^2=\left(\frac{a-b}{a+b}\right)^2\left[\frac{k^2+(a+b)^2}{k^2+(a-b)^2}\right]. \quad . \quad . \quad (419.)$$

When $k=\infty$, or when the paraboloid is a plane, $a_{\prime}=(a+b)$, $b_{\prime}=2\sqrt{ab}$, which are the values of the semiaxes of a plane ellipse, whose eccentricity is $\frac{a-b}{a+b}=\frac{1-\sqrt{1-i^2}}{1+\sqrt{1-i^2}}$, as we should have anticipated, for these are the values found in LXXVII. and LXXIX. for the axes of the derived plane ellipse.

When $m=n=1-j$, $m_{\prime}=\frac{mn}{(1+j)^2}=\left(\frac{1-j}{1+j}\right)^2=i_{\prime}^2$, and $n_{\prime}=0$.

Hence, when the original logarithmic ellipse is of the circular form, the first derived ellipse is a plane ellipse.

When $k^2=4ab$, (418.) shows that $m_{\prime}=n_{\prime}$, or $\frac{a_{\prime}}{k}=\frac{b_{\prime}}{k}=\infty$, as in XLIII.; but $m_{\prime}=n_{\prime}$, is equivalent to $n=m(\sqrt{1+j}+\sqrt{j})^2$.

Whenever therefore this relation exists between the parameters and modulus of the original integral, the first derived integral will represent the circular logarithmic ellipse, which may be integrated by functions of the first and second orders. Accordingly whenever the above relation exists between the parameters, the integral of the third order may be reduced to others of the first and second orders.

If in the second, third, or any other of the derived logarithmic ellipses, we can make the parameters equal, this derived ellipse will be of the circular form, and its rectification may be effected by integrals of the first and second orders only; accordingly the rectification of all the ellipses which precede it in the scale, may be effected by integrals of the first and second orders only.

We may repeat the remark made in LXXIX. The derived functions of two integrals of the logarithmic form with reciprocal parameters, have themselves reciprocal parameters.

LXXXII. If we now add together (162.) and (163.), we shall have

$$\left.\begin{aligned}\frac{4(n-m)}{\sqrt{mn}}\frac{\Sigma}{k}=&-\left[n\Phi_n+m\Phi_m\right]+\left[\frac{i^2}{m}+\frac{i^2}{n}-2\right]\int\frac{d\phi}{\sqrt{I}}\\&-(n-m)\left[\frac{(1-m)}{m}\int\frac{d\phi}{M\sqrt{I}}-\left(\frac{1-n}{n}\right)\int\frac{d\phi}{N\sqrt{I}}\right]+2\int d\phi\sqrt{I}-2\frac{(n-m)}{\sqrt{mn}}\int\frac{d\tau}{\cos^3\tau}\end{aligned}\right\}. \quad . \quad . \quad (420.)$$

We must now reduce this equation into functions of ψ instead of ϕ; ψ and ϕ being connected, as before, by the fundamental equation

$$\tan(\psi-\phi)=j\tan\phi.$$

The elements of these transformations are given at page 358, namely,

$$2\sin^2\phi = 1 + i_{,}\sin^2\psi - \cos\psi\sqrt{\mathrm{I}_{,}}, \text{ and } \frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}} = \sqrt{n_{,}}\sin\psi.$$

From this last equation we derive $(1-n\sin^2\phi)(1-m\sin^2\phi)=\mathrm{I}(1-m_{,}\sin^2\psi)$.

Now as $\Phi_n = \frac{\sin\phi\cos\phi\sqrt{\mathrm{I}}}{1-n\sin^2\phi}$, we shall have $n\Phi_n = \frac{\sqrt{m_{,}}\sin\psi}{2\sqrt{mn}}\left[\frac{2n-2nm\sin^2\phi}{1-m_{,}\sin^2\psi}\right]$. . . . (421.)

Or putting for $\sin^2\phi$ its value, $n\Phi_n = \frac{\sqrt{m_{,}}\sin\psi}{2\sqrt{mn}}\,\frac{[2n-nm-nmi_{,}\sin^2\psi+mn\cos\psi\sqrt{\mathrm{I}_{,}}]}{[1-m_{,}\sin^2\psi]}$. (422.)

In the same manner, we may find

$$m\Phi_m = \frac{\sqrt{m_{,}}\sin\psi}{2\sqrt{mn}}\,\frac{[2m-mn-mni_{,}\sin^2\psi+mn\cos\psi\sqrt{\mathrm{I}_{,}}]}{[1-m_{,}\sin^2\psi]}. \qquad (423.)$$

Adding those equations together, and recollecting that $m+n-mn=i^2$, we shall get

$$n\Phi_n + m\Phi_m = \frac{\sqrt{m_{,}}\,i^2\sin\psi}{\sqrt{mn}} + \frac{\sqrt{m_{,}}\sqrt{mn}\cos\psi\sin\psi\sqrt{\mathrm{I}_{,}}}{[1-m_{,}\sin^2\psi]}. \qquad (424.)$$

Now as $\qquad i^2=(1+j)(1-j)$, and $\sqrt{mn}=\sqrt{m_{,}}(1+j)$

$$-\{n\Phi_n+m\Phi_m\} = -(1-j)\sin\psi - (1+j)\frac{m_{,}\sin\psi\cos\psi\sqrt{\mathrm{I}_{,}}}{(1-m_{,}\sin^2\psi)}. \qquad (425.)$$

In (186.) we found

$$2\int d\phi\sqrt{\mathrm{I}} = (1+j)\int d\psi\sqrt{\mathrm{I}_{,}} - \frac{2j}{1+j}\int\frac{d\psi}{\sqrt{\mathrm{I}_{,}}} + (1-j)\sin\psi. \qquad (426.)$$

Adding this expression to the preceding, the terms involving $\sin\psi$ will disappear.

We must now compute the sum of the coefficients of $\int\frac{d\psi}{\sqrt{\mathrm{I}_{,}}}$.

Since $\quad \int\frac{d\phi}{\sqrt{\mathrm{I}}} = \frac{(1+i_{,})}{2}\int\frac{d\psi}{\sqrt{\mathrm{I}_{,}}}$, this coefficient becomes $\frac{(1+i)}{2}\left[\frac{i^2}{m}+\frac{i^2}{n}-2(1+j)\right]$.

Or as $m+n=i^2+mn$, this coefficient may be written $\left[\frac{i^4}{mn}+i^2-2(1+j)\right]\frac{(1+i_{,})}{2}$.

Or as $mn=m_{,}(1+j)^2$, it becomes finally, $\quad \frac{2}{1+i_{,}}\left[\frac{i_{,}^2}{m_{,}}-1\right]$. . . . . (427.)

Hence $\qquad \left[\frac{i^2}{m}+\frac{i^2}{n}-2(1+j)\right]\left(\frac{1+i_{,}}{2}\right)\int\frac{d\psi}{\sqrt{\mathrm{I}_{,}}} = \frac{2}{1+i_{,}}\left[\frac{i_{,}^2}{m_{,}}-1\right]\int\frac{d\psi}{\sqrt{\mathrm{I}_{,}}}$. . . . . . (428.)

And $(n-m)\left[\left(\frac{1-m}{m}\right)\int\frac{d\phi}{\mathrm{M}\sqrt{\mathrm{I}}} - \frac{(1-n)}{n}\int\frac{d\phi}{\mathrm{N}\sqrt{\mathrm{I}}}\right] = \frac{(n-m)}{\sqrt{mn}}\,\frac{(n-m)}{\sqrt{mn}}\,\frac{1}{(1+j)}\int\frac{d\psi}{[1-m_{,}\sin^2\psi]\sqrt{\mathrm{I}_{,}}}$. (429.)

Now as $\qquad n+m=i^2-mn,\ (n+m)^2=i^4-2mni^2+m^2n^2.$

Hence $\qquad (n-m)^2=i^4+2mni^2+m^2n^2-4mn,$

and as $\qquad i^4=(1+j)^2(1-j)^2,\ mn=m_{,}(1+j)^2$, substituting

$$(n-m)^2=(1+j)^2(1-j)^2+2m_{,}(1+j)^3(1-j)+m_{,}^2(1+j)^4-4m_{,}(1+j)^2,$$

therefore $$(n-m)^2=(1+j)^4\left[\left(\frac{1-j}{1+j}\right)^2+2m_i\left(\frac{1-j}{1+j}\right)+m_i^2-\frac{4m_i}{(1+j)^2}\right],$$

and as $\frac{4}{(1+j)^2}=(1+i_i)^2$, the expression will finally become

$$n-m=(1+j)^2(1-m_i)\sqrt{n_i}, \text{ hence } \frac{n-m}{\sqrt{mn}}\frac{i_i}{\sqrt{i}}=\left(\frac{1-m_i}{m_i}\right)\sqrt{m_in_i}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (430.)$$

If now we add together (420.), (425.), (426.), (428.) and (429.), we shall have, dividing by $\frac{(n-m)}{\sqrt{mn}}$,

$$\left.\begin{aligned}\frac{4\Sigma}{k}=-\frac{m_i\sqrt{m_i}}{(1-m_i)\sqrt{n_i}}\Phi_{m_i}+\frac{\sqrt{m_i}}{(1-m_i)\sqrt{n_i}}\int d\psi\sqrt{\mathrm{I}_i}\\ -\left(\frac{1-m_i}{m_i}\right)\sqrt{m_in_i}\int\frac{d\psi}{\mathrm{M}_i\sqrt{\mathrm{I}_i}}+\sqrt{\frac{n_i}{m_i}}\int\frac{d\psi}{\sqrt{\mathrm{I}_i}}-2\int\frac{d\tau}{\cos^3\tau}\end{aligned}\right\}. \quad . \quad . \quad . \quad (431.)$$

Let us now take the logarithmic ellipse whose equation contains m_i, n_i, i_i, ψ instead of m, n, i and φ, we shall have from (163.),

$$\begin{aligned}\frac{2\Sigma'}{k_i}=-\frac{m_i\sqrt{m_in_i}}{n_i-m_i}\Phi_{m_i}-\left(\frac{1-m_i}{m_i}\right)\sqrt{m_in_i}\int\frac{d\psi}{\mathrm{N}_i\sqrt{\mathrm{I}_i}}\\ +\frac{\sqrt{m_in_i}}{n_i-m_i}\int d\psi\sqrt{\mathrm{I}_i}+\frac{n_i}{m_i}\left(\frac{1-m_i}{n_i-m_i}\right)\sqrt{m_in_i}\int\frac{d\psi}{\sqrt{\mathrm{I}_i}}-2\int\frac{d\tau_i}{\cos^3\tau_i}.\end{aligned} \quad . \quad . \quad . \quad . \quad (432.)$$

If we now subtract these equations one from the other, combining together like integrals, the integral of the third order will vanish and we shall have,

$$\frac{2\Sigma_i}{k}-\frac{4\Sigma}{k}=\frac{m_i(1-n_i)\sqrt{m_in_i}}{n_i(n_i-m_i)(1-m_i)}\left[\int d\psi\sqrt{\mathrm{I}_i}+\frac{n_i}{m_i}(1-m_i)\int\frac{d\psi}{\sqrt{\mathrm{I}_i}}-m_i\Psi\right]+2\int\frac{d\tau}{\cos^3\tau}-2\int\frac{d\tau_i}{\cos^3\tau_i}. \quad . \quad . \quad (433.)$$

Hence, as we may express an arc of a plane ellipse by an arc of a derived ellipse, an integral of the first order, and a right line—a known theorem—so we may extend this analogy and express an arc of a logarithmic ellipse by an arc of a derived logarithmic ellipse, by functions of the first and second orders, by an arc of a parabola and by a right line. The relations between the moduli and amplitudes are the same in both cases,

$$i_i=\frac{1-j}{1+j}, \text{ and } \tan(\psi-\varphi)=j\tan\varphi.$$

Let m_{ii}, n_{ii}, i_{ii}, ψ_i be derived from m_i, n_i, i_i, ψ, by the same law as these latter are derived from m, n, i, φ, namely,

$$i_i=\frac{1-j}{1+j}, \quad \tan(\psi-\varphi)=j\tan\varphi, \quad m_i=\frac{mn}{(1+j)^2}, \quad n_i=\left[\frac{\sqrt{1-m}-\sqrt{1-n}}{\sqrt{1-m}+\sqrt{1-n}}\right]^2,$$

and derive an arc of a third logarithmic ellipse, we shall have, putting A, B, C, D for the coefficients of the integrals, and Π for the parabolic arc,

$$\frac{2\Sigma_i}{k}-\frac{4\Sigma}{k}=\mathrm{A}\int d\psi\sqrt{\mathrm{I}_i}+\mathrm{B}\int\frac{d\psi}{\sqrt{\mathrm{I}_i}}-\mathrm{C}\Psi+\mathrm{D}\Pi,$$

$$\frac{2\Sigma_{ii}}{k}-\frac{4\Sigma_i}{k}=\mathrm{A}'\int d\psi_i\sqrt{\mathrm{I}_{ii}}+\mathrm{B}'\int\frac{d\psi_{ii}}{\sqrt{\mathrm{I}_{ii}}}\mathrm{C}-'\Psi_i+\mathrm{D}'\Pi_i.$$

Multiply the first of these equations by 2 and add them, Σ will be eliminated. In this way we may successively eliminate $\Sigma_{,}$, $\Sigma_{,,}$, $\Sigma_{,,,}$, until ultimately we shall have

$$\frac{2\Sigma\nu}{k}-2^{\nu+1}\frac{\Sigma}{k}={}_{\nu}\mathrm{E}+{}_{\nu}\mathrm{F}+{}_{\nu}\overline{\Psi}-{}_{\nu}\overline{\Pi},$$

ν being the number of operations, and denoting by F and E, the sum of the integrals of the first and second orders, by $\overline{\Psi}$ the sum of the right lines, and by Π the sum of the parabolic arcs.

If in (401.) and (416.) we substitute the coefficients of the derived integrals as transformed in (404.) and (430.), the relation between the original and the derived integrals of the third order will be,

$$\left(\frac{1+n}{n}\right)\sqrt{mn}\int\frac{d\phi}{(1+n\sin^2\phi)\sqrt{1-i^2\sin^2\phi}}+\left(\frac{1-m}{m}\right)\sqrt{mn}\int\frac{d\phi}{(1-m\sin^2\phi)\sqrt{1-i^2\sin^2\phi}}=\left(\frac{1+n_{,}}{n_{,}}\right)\sqrt{m_{,}n_{,}}\int\frac{d\psi}{(1+n_{,}\sin^2\psi)\sqrt{1-i_{,}^2\sin^2\psi}}, \quad (434.)$$

for the circular form or the spherical ellipse, and

$$\left(\frac{1-m}{m}\right)\sqrt{mn}\int\frac{d\phi}{(1-m\sin^2\phi)\sqrt{1-i^2\sin^2\phi}}-\left(\frac{1-n}{n}\right)\sqrt{mn}\int\frac{d\phi}{(1-n\sin^2\phi)\sqrt{1-i^2\sin^2\phi}}=\left(\frac{1-m_{,}}{m_{,}}\right)\sqrt{m_{,}n_{,}}\int\frac{d\psi}{(1-m_{,}\sin^2\psi)\sqrt{1-i_{,}^2\sin^2\psi}}, \quad (435.)$$

for the logarithmic form, or the logarithmic ellipse.

LXXXIII. There are several plane curves, whose lengths we may express by elliptic integrals of the third order. For example, the length of the elliptic lemniscate, or the locus of the intersections of central perpendiculars on tangents to an ellipse, is equal to that of a spherical ellipse, which is supplemental to itself, or the sum of whose principal arcs is equal to π. We cannot represent elliptic integrals of the third order generally, by the arcs of curves, whose equations in their simplest forms contain only two constants. Thus let a and b be the constants. We shall have two equations between the constants the parameter and the modulus of the function, $i=f(a, b)$, $n=f'(a, b)$. Assume a as invariable, and eliminate b, we shall have one resulting equation between i, n, and a, or $F(a, i, n)=0$; or n depends on i.

When there are three independent constants, as in the preceding investigations, a, b, and k, we shall have $i=f(a, b, k)$, $n=f'(a, b, k)$. Eliminating successively b and k, we shall have two resulting equations, instead of one, $F(a, k, i, n)=0$, and $F'(a, b, i, n)=0$, or i and n depend on two equations, and may therefore be independent.

ERRATA.

Page 319, last line, *dele* n.
—— 320, line 5, *for* page 6 *read* page 316.
—— 328, line 12, *for* (47.) *read* (46.).
—— 329, line 15, *for* (32.) *read* (31.).
—— 331, line 7 from bottom, *for* $n=m=i$, *read* $n=m=\frac{1-j}{1+j}$.
—— 338, to the last line *add*, i being here the eccentricity of the base of the elliptic cylinder.
—— 372, line 11, *for* Case XII. *read* Case XIII.
—— 389, line 14, *for* BERNOUILLI *read* BERNOULLI.